

A STRONG DESINGULARIZATION THEOREM

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ABSTRACT. Let X be a closed subscheme embedded in a scheme W , smooth over a field \mathbf{k} of characteristic zero, and let $\mathcal{I}(X)$ be the sheaf of ideals defining X . Assume that the set of regular points of X is dense in X . We prove that there exists a proper, birational morphism, $\pi : W_r \longrightarrow W$, obtained as a composition of monoidal transformations, so that if $X_r \subset W_r$ denotes the strict transform of $X \subset W$ then:

1) The morphism $\pi : W_r \longrightarrow W$ is an embedded desingularization of X (as in Hironaka's Theorem);

2) The *total transform* of $\mathcal{I}(X)$ in \mathcal{O}_{W_r} factors as a product of an invertible sheaf of ideals \mathcal{L} supported on the exceptional locus, and the sheaf of ideals defining the strict transform of X (i.e. $\mathcal{I}(X)\mathcal{O}_{W_r} = \mathcal{L} \cdot \mathcal{I}(X_r)$).

This result is stronger than Hironaka's Theorem, in fact (2) is novel and does not hold for desingularizations which follow Hironaka's line of proof unless X is a hypersurface. We will say that $W_r \longrightarrow W$ defines a *Strong Desingularization* of X .

INTRODUCTION

After Hironaka's monumental work, where a non-constructive, existential proof of resolution of singularities was given (cf. [Hi1]), several alternative proofs have appeared. Some of them are simplified weak non-constructive versions (cf. [AJ], [AW], [BP]), while others are constructive algorithms of resolution of singularities (cf. [BM], [V1], [V2], [EV1]). The latest ones provide proofs of desingularization within Hironaka's line of development.

A simple alternative proof, conceptually different from Hironaka's, was pointed out in an addendum in [EV2] (see also [EV3]). There desingularization is achieved as a byproduct of a much simpler problem, namely that of *principalization of ideals*. Hilbert Samuel functions and normal flatness are avoided in the proof of desingularization with this strategy.

The alternative proofs of [EV2] and [EV3] rely on the fact that both desingularization and principalization are special cases of the so called *resolution of basic objects* (cf. Section 3). Recently G. Bodnár and J. Schicho have implemented an algorithm of resolution of basic objects in MAPLE, providing thereby a computer program of embedded desingularization (cf. [BS1] and [BS2]).

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Within the framework of this new generation of algorithms where principalization and desingularization follow from suitable defined resolution of basic objects, we obtain the new result presented here. By using a different *algorithm of desingularization*, the form of desingularization we achieve is stronger than Hironaka's and the ones given by previous algorithms. Our result is stated as follows:

Theorem of Strong Embedded Desingularization. *Let W be a smooth scheme over a field of characteristic zero k , and let $X \subset W$ be a closed subscheme. Assume that $\text{Reg}(X)$, the set of regular points in X , is dense in X (e.g. if X is reduced). Then there exists a proper birational morphism obtained as a composition of monoidal transformations,*

$$\Pi_r : W_r \longrightarrow W,$$

so that if $X_r \subset W_r$ is the strict transform of X , and if E_r is the exceptional locus of Π_r , then locally at every point $y \in W_r$ there will be a factorization of the total transform of the form:

$$(0.0.1) \quad J_{\pi(y)} \mathcal{O}_{W_r, y} = \mathcal{L}_y (J'_y) \subset \mathcal{O}_{W_r, y}$$

where \mathcal{L}_y is principal and supported on the exceptional locus of π , J'_y is the weak transform of J at y , and the following conditions hold:

- a) *If $J_{\pi(y)} = \mathcal{O}_{W, \pi(y)}$, or if $J_{\pi(y)} \neq \mathcal{O}_{W, \pi(y)}$ and $J_{\pi(y)}$ is a regular prime ideal, then $\mathcal{O}_{W, \pi(y)} = \mathcal{O}_{W_r, y}$.*
- b) *If $J_{\pi(y)} \neq \mathcal{O}_{W, \pi(y)}$ and $J_{\pi(y)}$ is not a regular prime ideal, then there is a regular system of parameters $\{x_1, \dots, x_d\} \subset \mathcal{O}_{W_r, y}$ and an integer $e \geq 0$ such that:*

$$(0.0.2) \quad \mathcal{L}_y = x_{e+1}^{c_{e+1}} \cdot \dots \cdot x_k^{c_k}, \quad \text{with } e < k \leq d$$

and

$$(0.0.3) \quad J'_y = \langle x_1, \dots, x_e \rangle,$$

and J'_y defines the strict transform of X at y .

We will say that $W_r \longrightarrow W$ defines a *Strong Desingularization* of X .

An announcement of this result appears in [BV].

In Section 1 we will reformulate our theorem to clarify how it relates to the classical formulation of desingularization (see Theorem 1.2). The fact that older results do not yield factorizations with the local conditions stated in (b) will be seen by analyzing the example where $W = \mathbb{A}_{\mathbb{Q}}^3 = \text{Spec } \mathbb{Q}[x_1, x_2, x_3]$ and X is the irreducible curve defined by the ideal $\mathcal{I}(X) = \langle x_1, x_2 x_3 + x_2^3 + x_3^3 \rangle$ (see Section 2 and Example 6.1). Our Strong Desingularization Theorem is therefore a novel algebraic formulation of embedded desingularization, which extends the classical result about embedded hypersurfaces to subschemes of arbitrary

codimension (note that if X is a hypersurface and if $\Pi_r : W_r \longrightarrow W$ defines an embedded resolution of X , then the local condition of part (b) always holds).

In our development we do not consider the notion of strict transforms of ideals. Given an ideal defining a reduced subscheme, and a sequence of monoidal transformations, we study the primary decomposition of its total transform, and we show that we can define a sequence of monoidal transformations so that all the new (exceptional) primary components are locally principal (defining the invertible ideal \mathcal{L} which appears in (0.0.1)). Our result answers a questions formulated by A. Nobile, which was the starting point of this research.

If $X \subset W$ is a complete intersection then a resolution of the sheaf \mathcal{O}_X by free \mathcal{O}_W -modules can be defined in terms of a Koszul complex. A byproduct of our theorem is that such Koszul complex (resolution) can be lifted to the Koszul complex providing a *resolution* of \mathcal{O}_{X_r} (as \mathcal{O}_{W_r} -module), after a simple twisting by the invertible sheaf \mathcal{L} appearing in (iii) of Theorem 1.2 (cf. Corollary 2.1).

Idea of the proof of the Theorem of Strong Embedded Desingularization: In this paper the approach to desingularization is substantially different from that in earlier works. Let $J = \mathcal{I}(X)$ be as in the theorem. Even if J is not reduced, we are requiring that there is a *dense* open set of regular points in X . This is the traditional requirement on a subscheme in order to formulate the problem of desingularization. As a consequence the primary components of J corresponding to the minimal associated prime ideals should be reduced. Now we turn to the question on how the primary decomposition of J can be modified after a sequence of monoidal transformations on W . Given a sequence of monoidal transformations

$$W_k \longrightarrow W_{r-1} \longrightarrow \dots \longrightarrow W_1 \longrightarrow W_0 = W$$

we will consider the factorization,

$$J\mathcal{O}_{W_i} = \mathcal{L}_i J'_i, \quad \text{for } i = 1, \dots, k$$

where J'_i denotes the weak transform of J in W_i and \mathcal{L}_i is locally a monomial ideal (see (0.0.1)). The strategy of our proof is to show first that we can define a sequence of monoidal transformations

$$W_l \longrightarrow W_{l-1} \longrightarrow \dots \longrightarrow W_0,$$

so that $V((J'_l)_y)$ is locally included in a smooth hypersurface for every $y \in V(J'_l)$. Then we show that by applying more monoidal transformations

$$W_m \longrightarrow W_{m-1} \longrightarrow \dots \longrightarrow W_l,$$

we may require $V((J'_m)_y)$ to be locally included in a smooth subscheme of codimension 2 for every $y \in V(J'_m)$, and so on. This leads us to the notion of *embedded relative codimension* stated in Definition 4.3. This an algebraic local condition on the factorization in (0.0.1). If the embedded relative codimension is at least one, then $V((J'_i)_y)$ is locally included in a smooth subscheme of codimension 1. If the embedded relative codimension is at least two,

then $V((J'_i)_y)$ is locally included in a smooth subscheme of codimension two, and so on. Lemma 4.8 shows that we can define a sequence of monoidal transformations so that the locally embedded codimension of the weak transform coincides with the codimension of X in W . It is in the proof of this lemma where we make use of some properties of algorithmic principalization. Following this argument we provide already a simple geometric proof of Theorem 1.2 in 4.9 .

The paper is organized as follows: After the formulation of Theorem 1.2 in Section 1, Section 2 is devoted to explain and illustrate what makes this work different from previous approaches to desingularization. In Section 3 we review the notion of *basic object* and that of *resolution of basic objects*, which is closely related to that of principalization of ideals (see Remark 3.5). As pointed out above, principalization of ideals plays an important role in the proof of Theorem 1.2. In Section 4 we show how Theorem 1.2 follows from Lemma 4.8 and a simple geometric argument (see 4.9). In Section 5 we explain the relation between Lemma 4.8 and the two main invariants of algorithmic principalization (in particular Remark 5.16 is already a hint for the proof of Lemma 4.8). This relation is also illustrated with examples in Section 6. Lemma 4.8 is finally proved in section 7.

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1. FORMULATION OF THE MAIN THEOREM

We briefly explain the notions of *pairs* and *transformation of pairs* which are suitable for the formulation of both *Strong Embedded Desingularization* and *Principalization of Ideals* (cf. Theorem 1.2 and Definition 1.3 below).

Definition 1.1. [EV2]. Let W be a pure dimensional scheme, smooth over a field \mathbf{k} of characteristic zero, and let $E = \{H_1, \dots, H_r\}$ be a set of smooth hypersurfaces in W with normal crossings (i. e. $\cup_{i=1}^r H_i$ has normal crossings). The couple (W, E) is said to be a *pair*. A regular closed subscheme $Y \subset W$ is said to be *permissible* for the pair (W, E) if Y has normal crossings with E .

If $Y \subset W$ is permissible for a pair (W, E) , we define a *transformation of pairs* in the following way: Consider the blowing-up with center Y , $W \xleftarrow{\Pi} W_1$, and define $E_1 = \{H'_1, \dots, H'_r, H_{r+1}\}$, where H'_i denotes the strict transform of H_i , and H_{r+1} denotes $\Pi^{-1}(Y)$ the exceptional hypersurface in W_1 . Note that W_1 is smooth and that E_1 has normal crossings. We say that $(W, E) \longleftarrow (W_1, E_1)$ is a *transformation of the pair* (W, E) .

1.2. Main Theorem (*of Strong Embedded Desingularization*) *Let $(W_0, E_0 = \{\emptyset\})$ be a pair and let $X_0 \subset W_0$ be a closed subscheme defined by $\mathcal{I}(X_0) \subset \mathcal{O}_{W_0}$. Assume that the open set $\text{Reg}(X)$ of regular points is dense in X (e.g. if X is reduced). Then there exists a finite sequence of transformations of pairs*

$$(1.2.1) \quad (W_0, E_0) \longleftarrow \cdots \longleftarrow (W_r, E_r)$$

inducing a proper birational morphism $\Pi_r : W_r \longrightarrow W_0$, so that setting $E_r = \{H_1, \dots, H_r\}$ and letting $X_r \subset W_r$ be the strict transform of X_0 , we have that:

- (i) X_r is regular in W_r , and $W_r \setminus \cup_{i=1}^r H_i \simeq W_0 - \text{Sing}(X)$. In particular $\text{Reg}(X) \cong \Pi_r^{-1}(\text{Reg}(X)) \subset X_r$ via Π_r .
- (ii) X_r has normal crossings with $E_r = \cup_{i=1}^r H_i$ (the exceptional locus of Π_r).
- (iii) The total transform of the ideal $\mathcal{I}(X_0) \subset \mathcal{O}_{W_0}$ factors as a product of ideals in \mathcal{O}_{W_r} :

$$\mathcal{I}(X)\mathcal{O}_{W_r} = \mathcal{L} \cdot \mathcal{I}(X_r),$$

where now $\mathcal{I}(X_r) \subset \mathcal{O}_{W_r}$ denotes the sheaf of ideals defining X_r , and $\mathcal{L} = \mathcal{I}(H_1)^{a_1} \cdot \dots \cdot \mathcal{I}(H_r)^{a_r}$ is an invertible sheaf of ideals supported on the exceptional locus of Π_r .

Note that part (i) of Theorem 1.2 ensures that the only points of W_0 that will be modified by the morphism Π_r are the ones in $\text{Sing}(X_0)$. In fact, the proof of the Theorem 1.2, follows from the simpler problem of principalization.

Definition 1.3. Let $I \subset \mathcal{O}_W$ be a sheaf of ideals. A *principalization* of I is a proper birational morphism $W_1 \longrightarrow W$ such that $I\mathcal{O}_{W_1}$ is an invertible sheaf of ideals in W_1 . A *strong principalization* of I is a chain of transformations of pairs

$$(W_0, E_0 = \emptyset) = (W, E) \longleftarrow \dots \longleftarrow (W_r, E_r)$$

such that $W \longleftarrow W_r$ defines an isomorphism over the open subset $W \setminus V(I)$, and

$$\mathcal{L} = I\mathcal{O}_{W_r} = \mathcal{I}(H_1)^{c_1} \cdot \dots \cdot \mathcal{I}(H_s)^{c_s},$$

where $E' = \{H_1, H_2, \dots, H_s\}$ are regular hypersurfaces with normal crossings and for $i = 1, \dots, s$, $c_i \geq 1$. If $V(I)$ is of codimension ≥ 2 , this means that $E' = E_r$ and the total transform of I is locally spanned by a monomial supported on the exceptional locus of $\Pi_r : W_r \rightarrow W$.

2. TOTAL TRANSFORM VERSUS STRICT TRANSFORM

Our next goal is to explain the difference between the algorithm of desingularization that we present in Theorem 1.2 and the previous ones which follow from Hironaka's line of proof. To do so we will study the example of desingularization of a curve embedded in a three-dimensional space. This example together with the discussion that follows it already appear in [BV, Section 3]. We reproduce it here for the convenience of the reader.

Let $W_0 = A_{\mathbb{Q}}^3 = \text{Spec}(\mathbb{Q}[x_1, x_2, x_3])$ and consider the curve C defined by

$$\mathcal{I}(C) = \langle x_1, x_2x_3 + x_2^3 + x_3^3 \rangle.$$

Let $W_0 \longleftarrow W_1$ be the quadratic transformation at the origin, $H \subset W_1$ the exceptional divisor, and C_1 the *strict transform* of C . This defines an embedded desingularization of C , in the usual sense, since both (i) and (ii) of Theorem 1.2 hold.

A) (*On condition 1.2 (iii)*). Since the ideal $\mathcal{I}(C)$ has order 1 at the center of the quadratic transformation, the *total transform* of $\mathcal{I}(C)$, namely $\mathcal{I}(C)\mathcal{O}_{W_1}$, can be factored as a product, $\mathcal{I}(C)\mathcal{O}_{W_1} = \mathcal{I}(H)^1 \bar{\mathcal{J}}_1$ for some coherent ideal $\bar{\mathcal{J}}_1 \subset \mathcal{O}_{W_1}$ which *does not* vanish along H .

Note that $\mathcal{I}(C_1)$ is a primary component of $\bar{\mathcal{J}}_1$. However, $\bar{\mathcal{J}}_1 \subsetneq \mathcal{I}(C_1)$, and hence Theorem 1.2 (iii) does not hold. To see why, it is convenient to express both ideals in terms of conductors: By definition,

$$\bar{\mathcal{J}}_1 = (\mathcal{I}(C)\mathcal{O}_{W_1} : \mathcal{I}(H)^1).$$

On the other hand, the ideal of the strict transform is an increasing union

$$\mathcal{I}(C_1) = \cup_{k \geq 0} (\mathcal{I}(C)\mathcal{O}_{W_1} : \mathcal{I}(H)^k),$$

or, in other words, $\mathcal{I}(C_1) = (\mathcal{I}(C)\mathcal{O}_{W_1} : \mathcal{I}(H)^N)$ for N large enough, since

$$(\mathcal{I}(C)\mathcal{O}_{W_1} : \mathcal{I}(H)^k) \subset (\mathcal{I}(C)\mathcal{O}_{W_1} : \mathcal{I}(H)^{k+1}).$$

In this example $H \simeq P_{\mathbb{Q}}^2$ and C_1 cuts $\mathbb{P}_{\mathbb{Q}}^2$ transversally at two different points. Let $L \subset \mathbb{P}_R^2$ be the line defined by these two points, and let $\mathcal{I}(L) \subset \mathcal{O}_{W_1}$ be the ideal of $L(\subset W_1)$. Then

$$\bar{\mathcal{J}}_1 = (\mathcal{I}(C)\mathcal{O}_{W_1} : \mathcal{I}(H)) \subsetneq \mathcal{I}(C_1) = (\mathcal{I}(C)\mathcal{O}_{W_1} : \mathcal{I}(H)^2),$$

and looking at a suitable affine chart it follows that $\mathcal{I}(L)$ is a primary component of $\bar{\mathcal{J}}_1$, but (of course), not of $\mathcal{I}(C_1)$ (see Example 6.1 where the same example is studied with more detail). Therefore (iii) of Theorem 1.2 does not hold for the embedded desingularization defined by Π .

In Hironaka's line of proof the centers of monoidal transformations, chosen in accordance to the so called *standard basis*, are always *included in the strict transform* of the scheme, so only *quadratic transformations* are applied in the one dimensional case. In the case of our singular curve, the first monoidal transformation must be the one we have defined above, and any other center will have dimension zero. Now $\mathcal{I}(L)$ is a primary component of $\bar{\mathcal{J}}_1$ supported on L , which has dimension 1; so we will never eliminate $\mathcal{I}(L)$ as a primary component of the total transform of $\mathcal{I}(C)$ by blowing up closed points; hence (iii) will never hold for desingularizations of this curve that follow from Hironaka's proof.

In order to achieve (iii) one must blow up L (or some strict transform of L). Using the new algorithm that we propose, we first consider the quadratic transformation $\Pi : W_1 \rightarrow W_0$, and then we blow-up at the one dimensional scheme L . Since $L \subset H$, the first isomorphism in Theorem 1.2 (i) is preserved after such monoidal transformation.

B) (*On a question of complexity*). We think of a subscheme X of a smooth scheme W , at least locally, as a finite number of *equations* defining the ideal $\mathcal{I}(X)$. An algorithm of desingularization should provide us with:

- (1) A sequence of monoidal transformations over the smooth scheme W , say $W_n \longrightarrow W_{n-1} \longrightarrow \dots \longrightarrow W_1 \longrightarrow W_0 = W$ so that conditions (i) and (ii) Theorem 1.2 hold for the strict transform of X at W_n .
- (2) A pattern of manipulation of equations defining X , so as to obtain, at least locally at an open covering of W_n , equations defining the strict transform X_n of X at W_n .

So (2) indicates how the original equations defining X have to be treated at an affine open subset of W_n in order to obtain local equations defining X_n . While this is very complicated in Hironaka's line of proof, here it is a direct consequence of Theorem 1.2 (iii). In fact, for algorithms that follow Hironaka's proof, to get both (1) and (2) one must consider the *strict transform* of the ideal of the subscheme at each monoidal transformation. In that setting one has to choose a *standard basis* of the ideal, which is a system of generators of the ideal of the subscheme, but such choice of generators must be changed if the maximum Hilbert Samuel invariant drops in the sequence of monoidal transformations. All of these complications are avoided in our new proof, which simplifies both (1) and (2). The simplifications attained in (2) are illustrated by the following corollary of Theorem 1.2:

Corollary 2.1. *Under the assumptions and with the notation of Theorem 1.2, if X is a complete intersection, then the resolution of \mathcal{O}_X in terms of free \mathcal{O}_W -modules*

$$\dots \longrightarrow \wedge^k \mathcal{O}_W \longrightarrow \wedge^{k-1} \mathcal{O}_W \longrightarrow \dots,$$

induces the resolution of \mathcal{O}_{W_r}

$$\dots \longrightarrow \mathcal{L}^{-k} \wedge^k \mathcal{O}_{W_r} \longrightarrow \mathcal{L}^{-k+1} \wedge^{k-1} \mathcal{O}_{W_r} \longrightarrow \dots,$$

in terms of locally free \mathcal{O}_{W_r} -modules.

Proof: Assume that W is affine, and that $X \subset W$ is defined by the complete intersection ideal $I(X) = \langle f_1, f_2, \dots, f_r \rangle \subset \mathcal{O}_W$. A resolution of \mathcal{O}_X by free \mathcal{O}_W -modules can be defined in terms of a Koszul complex. This complex is defined by taking the tensor product of

$$C_i := 0 \longrightarrow \mathcal{O}_W \cdot e_i \longrightarrow \mathcal{O}_W \longrightarrow 0,$$

where each such complex C_i is defined by $e_i \longrightarrow f_i$, for $i = 1, \dots, r$.

In the setting of 1.2 we have that $\mathcal{O}_W \subset \mathcal{O}_{W_r}$ and that $f_i \in \mathcal{L} \subset \mathcal{O}_{W_r}$. In particular each C_i induces a complex

$$\underline{C}_i := 0 \longrightarrow \mathcal{L}^{-1} \cdot e_i \longrightarrow \mathcal{O}_{W_r} \longrightarrow 0$$

Finally note that Theorem 1.2 (iii) says that the tensor product of these defines a resolution of \mathcal{O}_{X_r} in terms of locally free \mathcal{O}_{W_r} -modules. \square

The curve of our example $C \subset W = \mathbb{A}_{\mathbb{Q}}^3$ is a complete intersection; note that the result in this corollary *will never* hold for a desingularization of this curve given within Hironaka's line of proof, since condition (iii) of our Theorem will fail as seen in **A**).

3. BASIC OBJECTS

To achieve our results we use the notions of *basic objects* and *resolution of basic objects* (Definitions 3.1 and 3.4, see also [EV2]). Both Embedded Desingularization and Strong Principalization of Ideals can be obtained from a resolution of suitably defined basic objects.

Definition 3.1. A *basic object* is a triple that consists of a pair (W, E) , an ideal $J \subset \mathcal{O}_W$ such that $(J)_\xi \neq 0$ for any $\xi \in W$, and a positive integer b . It is denoted by $(W, (J, b), E)$. If the dimension of W is d , then $(W, (J, b), E)$ is said to be a d -dimensional *basic object*.

Definition 3.2. The *singular locus* of a basic object is the closed set:

$$\text{Sing}(J, b) = \{\xi \in W \mid \nu_J(\xi) \geq b\} \subset W,$$

where $\nu_J(\xi)$ denotes the order of the ideal J at the point ξ .

Definition 3.3. A regular closed subscheme $Y \subset W$ is *permissible* for $(W, (J, b), E)$ if Y is permissible for the pair (W, E) and $Y \subset \text{Sing}(J, b)$. Given $Y \subset \text{Sing}(J, b)$ we define a *transformation of basic objects* in the following way: Let $E = \{H_1, \dots, H_r\}$, and consider the blowing-up with center Y ,

$$W \longleftarrow W_1.$$

Denote by $H_{r+1} \subset W_1$ the exceptional hypersurface. This blowing-up induces a transformation of pairs

$$(W, E) \longleftarrow (W_1, E_1)$$

as in Definition 1.1. If Y is irreducible and if c_1 is the order of J at the generic point of Y (i.e. $\nu_J(Y) = c_1 \geq b$), then there is an ideal $\overline{J}_1 \subset \mathcal{O}_{W_1}$ such that

$$(3.3.1) \quad J\mathcal{O}_{W_1} = I(H_{r+1})^{c_1} \overline{J}_1.$$

The ideal \overline{J}_1 is usually called the *weak transform* of J . Note here that \overline{J}_1 does not vanish along H_1 (i.e. $H_1 \not\subseteq V(\overline{J}_1)$). Under these assumptions we define the ideal

$$(3.3.2) \quad J_1 = I(H_1)^{c_1-b} \overline{J}_1$$

and we set

$$(W, (J, b), E) \longleftarrow (W_1, (J_1, b), E_1)$$

as the *transformation of the basic object* $(W, (J, b), E)$.

In general, given a sequence of transformations of basic objects

$$(3.3.3) \quad (W_0, (J_0, b), E_0) \longleftarrow \dots \longleftarrow (W_k, (J_k, b), E_k),$$

at centers Y_i , $i = 0, 1, \dots, k-1$, we obtain expressions

$$(3.3.4) \quad J_i = I(H_{r+1})^{a_1} \dots I(H_{r+i})^{a_i} \overline{J}_i$$

and

$$(3.3.5) \quad J_0 \mathcal{O}_{W_i} = I(H_{r+1})^{c_1} \dots I(H_{r+i})^{c_i} \overline{J}_i,$$

with $c_j > a_j \geq 0$ (we refer to [EV2, 4.8] for a precise description of these exponents). Thus we have the factorization

$$(3.3.6) \quad J_0 \mathcal{O}_{W_i} = \mathcal{M}_i J_i = \mathcal{L}_i \overline{J}_i$$

for suitable defined invertible ideals \mathcal{M}_i and \mathcal{L}_i .

Definition 3.4. Sequence 3.3.3 is a *resolution* of $(W_0, (J_0, b), E_0)$ if $\text{Sing}(J_k, b) = \emptyset$.

Remark 3.5. If (3.3.3) is a resolution of the basic object $(W_0, (J_0, b), E_0)$ then $W_k \rightarrow W_0$ defines an isomorphism over $W_0 \setminus V(J_0)$, and $J_0 \mathcal{O}_{W_k} = \mathcal{M}_k J_k$, where \mathcal{M}_k is an invertible sheaf of ideals, and J_k has no points of order $\geq b$ in W_k . In particular a Strong Principalization of a sheaf of ideals $I \subset \mathcal{O}_W$ follows by taking a resolution of a basic object $(W_0, (J_0, b), E_0)$ with $W_0 = W$, $J_0 = I$, $b = 1$ and $E_0 = \emptyset$.

A resolution of basic objects is usually approached by means of an *algorithm of resolution of basic objects*, and the factorization in (3.3.4) plays a central role in this procedure (see Section 5). To prove Theorem 1.2 we modify the algorithm of resolution of basic objects which appears in [EV1].

4. PROOF OF THEOREM 1.2

4.1. Let $(W_0, E_0 = \emptyset)$ and X be as in Theorem 1.2 and let $J_0 = \mathcal{I}(X) \subset \mathcal{O}_{W_0}$. To prove Theorem 1.2 we will define a suitable sequence of transformations of pairs at permissible centers

$$(4.1.1) \quad (W_0, E_0 = \emptyset) \leftarrow \dots \leftarrow (W_m, E_m),$$

together with expressions

$$(4.1.2) \quad J_0 \mathcal{O}_{W_i} = \mathcal{I}(H_1)^{c_1} \cdot \dots \cdot \mathcal{I}(H_i)^{c_i} \overline{J}_i$$

as in (3.3.5), so that the conclusion of Theorem 1.2 holds for

$$J_0 \mathcal{O}_{W_m} = \mathcal{I}(H_1)^{c_1} \cdot \dots \cdot \mathcal{I}(H_k)^{c_m} \overline{J}_m.$$

First we will motivate the argument of the proof of Theorem 1.2 from the algebraic point of view. This will lead us to Definition 4.3 and to the statement of Lemma 4.8. The proof of Theorem 1.2 is presented in 4.9.

The proof of Lemma 4.8 will be given in Section 7. In order to prove this lemma we will need some extra ingredients which will be presented in Section 5. In Section 6 we exhibit some examples illustrating the main ideas of Section 5.

4.2. Conditions on (4.1.1) and (4.1.2): The embedded relative codimension. Let $(W, E = \{H_1, \dots, H_s\})$ be a pair and let $y \in W$ be a point. Then we can set a regular system

of parameters $\{x_1, \dots, x_n\} \subset \mathcal{O}_{W,y}$, so that if $y \in \cup_{j=i_1}^{i_r} H_{i_j}$, then $\cup_{j=i_1}^{i_r} H_{i_j}$ is locally defined by

$$(4.2.1) \quad x_{i_1} \cdot \dots \cdot x_{i_r} = 0$$

at a neighborhood of y in W .

Assume that (4.1.1) is defined so that at any point $y \in V(\bar{\mathcal{J}}_m) \subset W_m$, $(\bar{\mathcal{J}}_m)_y$ is an ideal of order 1 in the local regular ring $\mathcal{O}_{W_m,y}$. In such case we may choose a regular system of parameters $\{z_1, \dots, z_n\} \subset \mathcal{O}_{W_m,y}$ so that

$$(4.2.2) \quad z_1 \in (\bar{\mathcal{J}}_m)_y.$$

The proof of Theorem 1.2 is based in the idea of defining suitable sequences of transformation of basic objects

$$(4.2.3) \quad (W_0, E_0 = \emptyset) \longleftarrow \dots \longleftarrow (W_k, E_k),$$

such that at any point $y \in V(\bar{\mathcal{J}}_k) \subset W_k$, there is a regular system of parameters so that both conditions (4.2.1) and (4.2.2) are simultaneously satisfied. This leads to the following definition:

Definition 4.3. Let (W, E) be a pair, and let $I \subset \mathcal{O}_W$ be a non-zero sheaf of ideals.

(1) We shall say that I has *relative embedded local codimension* $\geq a$ at a point $y \in W$, if either $I_y = \mathcal{O}_{W,y}$, or there is a regular system of parameters $\{x_1, x_2, \dots, x_n\}$ at $\mathcal{O}_{W,y}$, such that:

- (i) $\langle x_1, x_2, \dots, x_a \rangle \subset I_y \subset \mathcal{O}_{W,y}$, and
- (ii) any hypersurface $H_i \in E$ containing the point y has a local equation

$$\mathcal{I}(H_i) = \langle x_{i_j} \rangle \subset \mathcal{O}_{W,y},$$

with $i_j > a$.

(2) We shall say that I has *relative embedded codimension* $\geq a$ in (W, E) , if both conditions in (1) hold at every point $y \in W$. We will abbreviate this saying that I has *relative codimension* $\geq a$ in (W, E) .

Remark 4.4. Note that if I is of relative codimension $\geq a$, then the closed subscheme $V(I)$ is in fact of codimension $\geq a$ in W . Note also that any non-zero ideal $I \subset \mathcal{O}_W$ is of relative codimension ≥ 0 in (W, E) since such condition is empty.

Remark 4.5. If $X \subset W$ is a regular scheme of pure codimension e , then the sheaf of ideals $\mathcal{I}(X) \subset \mathcal{O}_W$ has relative codimension $\geq e$ in $(W, E = \emptyset)$.

Remark 4.6. Let $(W_i, (J_i, b), E_i)$ be a basic object and let $J_i = \mathcal{L}_i \bar{J}_i$ be as in (3.3.4). If \bar{J}_i has relative codimension $\geq a$ in (W, E) , then for every $y \in V(\bar{J}_i)$ there is a regular system of parameters $\{x_1, \dots, x_n\} \subset \mathcal{O}_{W_i, y}$ such that

$$(4.6.1) \quad J\mathcal{O}_{W_i, y} = x_{i_1}^{c_1} \cdot \dots \cdot x_{i_r}^{c_r} (\bar{J}_i)_y; \quad \langle x_1, \dots, x_a \rangle \subset (\bar{J}_i)_y$$

with $a < i_1 < i_2 < \dots < i_r \leq n$ and $(\mathcal{L}_i)_y = \langle x_{i_1}^{c_1} \cdot \dots \cdot x_{i_r}^{c_r} \rangle$.

Definition 4.7. With the same notation as in Remark 4.6, we will say that \bar{J}_i is of (W_i, E_i) -codimension $\geq a$ if \bar{J}_i has relative codimension $\geq a$ in (W_i, E_i) .

Now assume that X is under the assumptions of Theorem 1.2, and that it has codimension e as subscheme in W . Then the local codimension of $J_0 = I(X)$ is $\geq e$ at every point of $V_0 = W_0 \setminus \text{Sing}(X)$. Therefore J_0 will be of (W_0, E_0) -codimension $\geq a$ for some $a \leq e$ (since by Remark 4.4 we may always assume $a = 0$). Now Theorem 1.2 will follow from Lemma 4.8 and the argument given in 4.9.

Lemma 4.8. *Let X be under the assumptions of Theorem 1.2, let*

$$(W_0, (J_0, 1), E_0) = (W, (\mathcal{I}(X), 1), \emptyset)$$

and assume that there is a sequence of transformations of pairs

$$(4.8.1) \quad (W_0, E_0) \longleftarrow \dots \longleftarrow (W_k, E_k)$$

such that

$$\mathcal{I}(X)\mathcal{O}_{W_k} = \mathcal{L}_k \bar{J}_k$$

and that \bar{J}_k is of (W_k, E_k) -codimension $\geq a$. Let $U_k \subset W_k$ be a non-empty open set such that for every $y \in U_k$, the local codimension of $(\bar{J}_k)_y$ is $\geq a + 1$. Then we can define an enlargement of the sequence 4.8.1,

$$(4.8.2) \quad (W_k, E_k) \longleftarrow \dots \longleftarrow (W_N, E_N)$$

so that:

- (i) $\mathcal{I}(X)\mathcal{O}_{W_N} = \mathcal{L}_N \bar{J}_N$ and \bar{J}_N is of (W_N, E_N) -codimension $\geq a + 1$.
- (ii) The birational morphism $W_k \longleftarrow W_N$ defines an isomorphism over the open set $U_k \subset W_k$.

4.9. Proof of Theorem 1.2. Let $X \subset W$ be a subscheme of codimension e , under the assumptions of Theorem 1.2. Set $W_0 = W$, $J_0 = \mathcal{I}(X)$, $E_0 = \emptyset$, and $U_0 = W_0 \setminus \text{Sing}(X)$. Then $U_0 \subset W$ is a dense open subset, $U_0 \cap X$ is dense in X , and X is regular at all points of $U_0 \cap X$. Since $E_0 = \emptyset$, the ideal J_0 is of relative local codimension $\geq e$ at every point $x \in U_0 \cap X$. In particular, there is an integer a , $0 \leq a \leq e$, so that J_0 has (W_0, E_0) -codimension $\geq e$ (see Remark 4.4). If $a < e$, by successive applications of Lemma 4.8, we may define a sequence of transformations

$$(4.9.1) \quad (W_0, E_0) \longleftarrow \dots \longleftarrow (W_{N_e}, E_{N_e})$$

so that if

$$(4.9.2) \quad I(X)\mathcal{O}_{W_{N_e}} = \mathcal{L}_{N_e}\bar{J}_{N_e}$$

then

- (I) \bar{J}_{N_e} has (W_{N_e}, E_{N_e}) -codimension $\geq e$.
- (II) $W_0 \longleftarrow W_{N_e}$ defines an isomorphism over U_0 , say $U_0 \simeq U_{N_e} \subset W_{N_e}$.

First we will make some elementary geometric remarks on the closed set $V(\bar{J}_{N_e}) \subset W_{N_e}$:

- (a) $V(\bar{J}_{N_e})$ has codimension $\geq e$ as closed subscheme of W_{N_e} . Let

$$(4.9.3) \quad V(\bar{J}_{N_e}) = F_1 \cup \dots \cup F_l \cup C_{l+1} \cup \dots \cup C_p$$

be the union of irreducible components (each of codimension at least e in W_{N_e}), where the F_i are the components of codimension e . Set $F = F_1 \cup \dots \cup F_l$.

- (b) Each component F_i is a smooth and *connected component* of $V(\bar{J}_{N_e})$.
- (c) $V(\bar{J}_{N_e}) \cap U_{N_e} \simeq X \setminus \text{Sing}(X) = X \cap U_0$ (U_{N_e} and U_0 as in (II)).

Conditions (a) and (b) can be checked from Definition 4.3. Condition (c) follows from (II). If we assume that X is of *pure* codimension e , then (b) and Definition 4.3 assert that:

- (d) If $X_{N_e} \subset W_{N_e}$ denotes the strict transform of X , then

$$X_{N_e} = F_1 \cup \dots \cup F_{l'}, \quad l' \leq l$$

is a disjoint union of closed regular subschemes.

Condition (ii) in Definition 4.3 asserts that conditions (i) and (ii) of Theorem 1.2 hold for $W_{N_e} \longrightarrow W_0$. Now by (d) we conclude that:

- (e) For every $y \in X_{N_e} = F_1 \cup \dots \cup F_{l'}$,

$$(\bar{J}_{N_e})_y = (\mathcal{I}(X_{N_e}))_y \subset \mathcal{O}_{W_{N_e}, y}.$$

In fact, we know that $I(X_{N_e})_y$ is a primary component of $(\bar{J}_{N_e})_y$, so $(\bar{J}_{N_e})_y \subset (\mathcal{I}(X_{N_e}))_y$; if this inclusion were proper, then $V(\bar{J}_{N_e})$ would have codimension $> e$ locally at y (but $y \in F \subset V(J_{N_e})$). Note that (e) is saying that condition (iii) of Theorem 1.2 holds locally at each point in $F_1 \cup \dots \cup F_{l'}$.

Since $V(\bar{J}_{N_e})$ is a *disjoint union* of $F_1 \cup \dots \cup F_{l'}$ and $F_{l'+1} \cup \dots \cup F_l \cup C_1 \cup \dots \cup C_p$, we can express \bar{J}_{N_e} as a product of two ideals, $\bar{J}_{N_e} = [A_{N_e}]_1 \cdot [A_{N_e}]_2$, so that

$$V([A_{N_e}]_1) = F_1 \cup \dots \cup F_{l'} \text{ and } V([A_{N_e}]_2) = F_{l'+1} \cup \dots \cup F_l \cup C_1 \cup \dots \cup C_p.$$

Note finally that $V([A_{N_e}]_2) \subset W_{N_e} \setminus V([A_{N_e}]_1)$. If X is of pure codimension e , then (iii) of Theorem 1.2 is finally achieved by setting

$$(W_{N_e}, E_{N_e}) \longleftarrow \dots \longleftarrow (W_r, E_r)$$

so as to define a principalization of $[A_{N_e}]_2 \subset \mathcal{O}_{W_{N_e}}$.

If X is non-pure-dimensional the proof follows in a rather similar fashion: By (b) we have that at $W_{N_e} \setminus V([A_{N_e}]_1)$ the ideal $[A_{N_e}]_2$ factors as a product of two sheaves of ideals

$$(4.9.4) \quad [A_{N_e}]_2 = [A_{N_e}]_3 \cdot [A_{N_e}]_4,$$

so that

$$(4.9.5) \quad V([A_{N_e}]_4) = C_{l+1} \cup \dots \cup C_p$$

where $\dim(C_i) \geq e + 1$, for $i = l + 1, \dots, p$.

Now $X_{N_e} \cap [W_{N_e} \setminus V([A_{N_e}]_1)]$ is a union of some of the irreducible components C_i of $V([A_{N_e}]_4)$ appearing in (4.9.5). These components are regular in a dense open set. Then we apply Lemma 4.8 for some $e' > e$ and make use of (b) to separate those e' -dimensional components of $V([A_{N_e}]_4)$ which are components of the strict transform of X from those which are not. Thus we repeat the same argument that we applied in the pure codimensional case, but now for higher codimensions. \square

5. ALGORITHMS OF RESOLUTION OF BASIC OBJECTS

Lemma 4.8 will follow from a suitable Algorithm of Resolution of Basic Objects. An Algorithm of Resolution of Basic Objects is defined by means of an *upper semi-continuous function* (see Definition 5.2). In 5.14 we will briefly describe these upper semi-continuous functions to give an idea of the strategy that we will follow (see also [EV2]). The novelty of our development appears in 5.16 and 5.17, where we show how the two main invariants involved in the algorithm of resolution of basic objects (namely in the definition of the upper semi-continuous function) relate to our notion of relative codimension stated in Definition 4.3.

Definition 5.1. let X be a topological space, and let (T, \geq) be a totally ordered set. Let $g : X \longrightarrow T$ be an upper semi-continuous function, and assume that g takes only finitely many values. The largest value achieved by g will be denoted by

$$\max g.$$

Clearly the set

$$\underline{\text{Max}} g = \{x \in X : g(x) = \max g\}$$

is a closed subset of X .

Definition 5.2. Let d be a non-negative integer. An *algorithm of resolution for d -dimensional basic objects*, consists of:

- A. A totally ordered set (I_d, \leq) .
- B. For each d -dimensional basic object $(W_0, (J_0, b), E_0)$ (with E_0 is not necessarily empty), an upper semi-continuous function

$$f_0^d : \text{Sing}(J_0, b) \longrightarrow I_d$$

such that $\underline{\text{Max}} f_0^d$ is permissible for $(W_0, (J_0, b), E_0)$.

C. Suppose that we define a sequence of blowing-ups at permissible centers $Y_i \subset \text{Sing}(J_i, b)$, $i = 0, \dots, r-1$,

$$(5.2.1) \quad (W_0, (J_0, b), E_0) \longleftarrow \cdots \longleftarrow (W_{r-1}, (J_{r-1}, b), E_{r-1}) \longleftarrow (W_r, (J_r, b), E_r),$$

together with a sequence of upper semi-continuous functions

$$f_i^d : \text{Sing}(J_i, b) \longrightarrow I_d, \quad i = 0, \dots, r-1,$$

so that

$$Y_i = \underline{\text{Max}} f_i^d.$$

Then if $\text{Sing}(J_r, b) \neq \emptyset$, there is an upper semi-continuous function

$$f_r^d : \text{Sing}(J_r, b) \longrightarrow I_d$$

such that $\underline{\text{Max}} f_r^d$ is permissible for $(W_r, (J_r, b), E_r)$.

D. For some index r_0 , depending on the basic object $(W_0, (J_0, b), E_0)$, the sequence constructed in **C** is a resolution (i.e. $\text{Sing}(J_{r_0}, b) = \emptyset$).

E. Properties. The following properties should hold:

- i. If $\xi \in \text{Sing}(J_i, b)$ and $\xi \notin Y_i$ for $i = 0, \dots, r_0 - 1$, then $f_i^d(\xi) = f_{i+1}^d(\xi')$ via the natural identification of the point ξ with a point ξ' of $\text{Sing}(J_{i+1}, b)$.
- ii. The resolution is obtained by transformations with centers $\underline{\text{Max}} f_i^d$, for $i = 0, \dots, r_0 - 1$, and

$$\max f_0^d > \max f_1^d > \cdots > \max f_{r_0-1}^d$$

- iii. If J_0 is the ideal of a regular pure dimensional subscheme X_0 , if $E_0 = \emptyset$, and if $b = 1$, then the function f_0^d is constant.
- iv. For every $i = 0, \dots, r_0 - 1$, the closed set $\underline{\text{Max}} f_i^d$ is smooth, equidimensional, and its dimension is determined by the value $\max f_i^d$.

Remark 5.3. Note that **B** asserts that the setting of (5.2.1) holds for $r = 1$, whereas **C** says that whenever $\text{Sing}(J_r, b) \neq \emptyset$ there is an enlargement of (5.2.1) with center $Y_r = \underline{\text{Max}} f_r^d$. Property **E** (i) ensures that the algorithm commutes with open restrictions: If we restrict to a non empty open subset of W , then the restriction of the algorithm gives the resolution of the restriction of the basic object to the open subset. Property **E** (ii) says that the heart of the matter in the algorithm is to find a good upper semi-continuous function that controls the singular locus of the basic object, determines the center to blow-up (see **B**, **C** and **E** (iv)) and guarantees that at each step the *singularities* are getting better (property **E** (ii)).

5.4. An algorithm of resolution of basic objects, as defined in 5.2 and with the properties stated in **E**, is presented in [EV2, 7.15]. There are several ways in which embedded desingularization can be achieved in terms of a fixed algorithm of resolution of basic objects. It is indicated in [EV2] (see the addendum) how a particular theorem of embedded desingularization can be easily deduced as a byproduct from the properties listed in **E**. In what follows we will briefly explain how the functions f_i^d are defined, and for more details the reader is referred to [EV2].

How to define the functions f_i^d .

Definition 5.5. Let $(W_0, (J_0, b), E_0 = \{H_1, \dots, H_l\})$ be a d -dimensional basic object. Given a sequence of permissible transformations,

$$(5.5.1) \quad (W_0, (J_0, b), E_0) \xleftarrow{\pi_1} \dots \xleftarrow{\pi_r} (W_r, (J_r, b), E_r)$$

together with the corresponding expressions

$$(5.5.2) \quad J_i = I(H_{l+1})^{a_1} \dots I(H_{l+i})^{a_r} \bar{J}_i,$$

where H_{l+i} denotes the exceptional divisor at that i -th blowing-up, we define the upper semi-continuous functions

$$\begin{aligned} w - ord_i^d : \text{Sing}(J_i, b) &\longrightarrow \frac{1}{b}\mathbb{Z} \subset \mathbb{Q} \\ \xi &\longrightarrow \frac{\nu_{\bar{J}_i}(\xi)}{b} \end{aligned}$$

Remark 5.6. Note that for any $k \in \mathbb{N}_{\geq 1}$ a resolution of $(W_0, (J_0, b), E_0)$ is *equivalent* to a resolution of $(W_0, (J_0^k, kb), E_0)$. In fact it suffices to take k -th powers in the factorization in (5.5.2). So it is reasonable to expect that the function $w - ord_i^d$ be compatible with this equivalence. This explains why the function $w - ord_i^d$ is defined taking the quotient by b in the previous definition.

Remark 5.7. Note that since $w - ord_i^d$ is upper semi-continuous the set

$$\text{Max } w - ord_i^d = \{\xi \in \text{Sing}(J_i, b) : w - ord_i^d(\xi) = \max w - ord\}$$

is closed in $\text{Sing}(J_i, b) \subset W_i$. Note also that $\max w - ord_i^d = 0$ if and only if $\bar{J}_i = \mathcal{O}_{W_i}$.

Remark 5.8. We will only consider sequences of transformations of basic objects at permissible centers Y_i such as the ones defined in (5.5.1), with the additional constraint that

$$(5.8.1) \quad Y_i \subset \text{Max } w - ord_i^d \subset \text{Sing}(J_i, b)$$

for $i = 0, 1, \dots, r$. In such case

$$(5.8.2) \quad w - ord_{i-1}^d(\pi_i(\xi_i)) \geq w - ord_i^d(\xi_i)$$

for every $\xi \in \text{Sing}(J_i, b)$, and equality holds if $\xi \notin Y_i$. In particular

$$(5.8.3) \quad \max w - ord_0^d \geq \dots \geq \max w - ord_r^d$$

(see [EV2, 4.12]).

Definition 5.9. Consider a sequence of transformations of basic objects as in (5.5.1), with

$$Y_i \subset \text{Max } w - ord_i^d$$

for $i = 0, 1, \dots, r$. Pick $k \in \{0, 1, \dots, r\}$. If $\max w - ord_0^d > 0$ let k_0 be the smallest index so that

$$\max w - ord_{k_0-1}^d > \max w - ord_{k_0}^d = \max w - ord_k^d$$

($k_0 = 0$ if $\max w - \text{ord}_0^d = \dots = \max w - \text{ord}_r^d$). Write

$$E_k = E_k^+ \sqcup E_k^-$$

where E_k^- is the set hypersurfaces of E_k which are strict transforms of hypersurfaces of E_{k_0} . Now define

$$n_k^d(\xi) = \begin{cases} \#\{H \in E_k \mid \xi \in H\} & \text{if } w - \text{ord}_k^d(\xi) < \max w - \text{ord}_k^d \\ \#\{H \in E_k^- \mid \xi \in H\} & \text{if } w - \text{ord}_k^d(\xi) = \max w - \text{ord}_k^d. \end{cases}$$

Definition 5.10. With the hypothesis of Definition 5.9, if $\max w - \text{ord}_r^d > 0$, we define, for the index r , a function t_r^d by setting:

$$\begin{aligned} t_r^d : \text{Sing}(J_r, b) &\longrightarrow (\mathbb{Q} \times \mathbb{Z}, \leq) && \text{(lexicographic order)} \\ \xi &\longrightarrow (w - \text{ord}_r^d(\xi), n_r^d(\xi)) \end{aligned}$$

In the same way we define functions $t_{r-1}^d, t_{r-2}^d, \dots, t_0^d$. If $Y_r \subset \underline{\text{Max}} t_r^d$ is a permissible center, then Y_r is said to be a t_r^d -permissible center.

5.11. Properties of the function t_i^d (cf. [EV2, 4.15]). The functions

$$t_i^d : \text{Sing}(J_i, b) \longrightarrow I_d$$

have the following properties:

1. Each $t_i^d : \text{Sing}(J_i, b) \longrightarrow \mathbb{Q} \times \mathbb{Z}$ is upper semi-continuous.
2. If in the sequence of transformations

$$(5.11.1) \quad \begin{aligned} &(W_0, (J_0, b), E_0) \xleftarrow{\pi_1} (W_1, (J_1, b), E_1) \xleftarrow{\pi_2} \dots \\ &\dots \xleftarrow{\pi_{r-1}} (W_{r-1}, (J_{r-1}, b), E_{r-1}) \xleftarrow{\pi_r} (W_r, (J_r, b), E_r) \end{aligned}$$

the permissible centers Y_i have the additional constrain

$$Y_i \subset \underline{\text{Max}} t_i^d \subset \underline{\text{Max}} w - \text{ord}_i \quad i = 0, 1, \dots, r-1$$

then for each index $i = 0, 1, \dots, r$,

$$t_{i-1}^d(\pi_i(\xi_i)) \geq t_i^d(\xi_i) \quad \text{for all } \xi_i \in \text{Sing}(J_i, b)$$

and equality holds if $\pi_i(\xi_i) \notin Y_{i-1}$. In particular

$$\max t_0^d \geq \dots \geq \max t_r^d.$$

3. We say that $\max t^d$ drops at i_0 if $\max t_{i_0-1}^d > \max t_{i_0}^d$. If $\max w - \text{ord}_0^d = \frac{b'}{b}$ and $\dim W_0 = d$, note that $\max t_i^d = \left(\frac{s}{b}, m\right)$, $0 \leq s \leq b'$, $0 \leq m \leq d$. So it is clear that $\max t^d$ can drop at most $b'd$ times.

4. The functions t_i^d are the *inductive invariant*: via some form of induction, which will be described in 5.14, it is possible to construct a unique enlargement of (5.11.1)

$$(5.11.2) \quad (W_0, (J_0, b), E_0) \longleftarrow \cdots \longleftarrow (W_r, (J_r, b), E_r) \longleftarrow \\ \longleftarrow (W_{r+1}, (J_{r+1}, b), E_{r+1}) \longleftarrow \cdots \longleftarrow (W_N, (J_N, b), E_N)$$

so that $\max t_r^d = \max t_{r+1}^d = \cdots = \max t_{N-1}^d$ and either

- (a) $\text{Sing}(J_N, b) = \emptyset$.
- (b) $\text{Sing}(J_N, b) \neq \emptyset$ and $\max w - \text{ord}_N = 0$.
- (c) $\text{Sing}(J_N, b) \neq \emptyset$, $\max w - \text{ord}_N > 0$ and $\max t_{N-1} > \max t_N$.

Note that Property 3 says that $\max w - \text{ord}^d$ can drop finitely many times, in particular for some index N , either (a) or (b) will hold.

Remark 5.12. We will consider here d -dimensional basic objects $(W_0, (J_0, b), E_0)$ together with sequences of t_i^d -permissible transformations,

$$(5.12.1) \quad (W_0, (J_0, b), E_0) \longleftarrow \cdots \longleftarrow (W_r, (J_r, b), E_r).$$

In such case $\max w - \text{ord}_i^d \geq \max w - \text{ord}_{i+1}^d$ and $\max t_i^d \geq \max t_{i+1}^d$. We shall also consider the factorization

$$(5.12.2) \quad J_i = \mathcal{I}(H_{l+1})^{a_1} \cdots \mathcal{I}(H_{l+i})^{a_i} \overline{J}_i$$

as in (3.3.4).

The condition $\max w - \text{ord}_i^d > 0$ is equivalent to $\overline{J}_i \subset \mathcal{O}_{W_i}$, and $\overline{J}_i \neq \mathcal{O}_{W_i}$. If $\max w - \text{ord}_i^d = 0$, then $\overline{J}_i = \mathcal{O}_{W_i}$ and a resolution of $(W_i, (J_i, b), E_i)$ is a simple combinatorial problem (cf. [EV2, Section 5]). If $\max w - \text{ord}_i^d > 0$, then

$$(5.12.3) \quad \underline{\text{Max}} t_i^d \subset \underline{\text{Max}} w - \text{ord}_i^d \subset V(\overline{J}_i).$$

Remark 5.13. Let $(W_0, (J_0, 1), E_0 = \{H_1, \dots, H_l\})$ be a d -dimensional basic object and consider a sequence of permissible transformations,

$$(5.13.1) \quad (W_0, (J_0, 1), E_0) \longleftarrow \cdots \longleftarrow (W_r, (J_r, 1), E_r)$$

together with the corresponding expressions,

$$(5.13.2) \quad J_i = I(H_{l+1})^{a_1} \cdots I(H_{l+i})^{a_r} \overline{J}_i,$$

then if $\max t_i^d = (1, 0)$, \overline{J}_i is of (W_i, E_i) -codimension ≥ 1 (see Remark 5.16 for the proof of this fact). Therefore, in our development we are particularly interested in resolutions of basic objects $(W, (J, b), E)$ with $b = 1$. However, the construction of such resolutions involves the definition of new basic objects in order to get suitable invariants, and then we will have to consider the case when $b > 1$ (see 5.14).

5.14. The functions t_i^d and the functions f_i^d . We will present the construction of the functions f_i^d in three steps, and we refer to [EV2] for more details. First we will give a brief idea of the strategy that we will follow: The main point is to give a precise formulation of the *inductive invariant* and the unique enlargement of the sequence (5.11.2). To this purpose, given a basic object $(W_i, (J_i, b), E_i)$ we will associate to it another $(W_i, (J_i'', b''), E_i'')$ which will be constructed by means of a suitable differential operator applied to J_i . Using this new basic object we can associate to $(W_i, (J_i, b), E_i)$ a new $d - 1$ dimensional basic object which will be described in step 2. This is the key point of induction on the dimension of the basic objects.

Although the definition of these functions is based in a local argument (see Steps 1 and 2) the construction is natural enough so as to produce globally defined functions (see Remark 5.15).

Step 1. Let $(W_i, (J_i, b), E_i)$ be a basic object, and let

$$t_i^d : \text{Sing}(J_i, b) \longrightarrow (\mathbb{Q} \times \mathbb{Z}, \leq),$$

be as in Definition 5.10. Consider the closed set $\underline{\text{Max}} t_i^d \subset \text{Sing}(J_i, b)$. Then for any $x \in \underline{\text{Max}} t_i^d$ there is a new basic object

$$(W_i, (J_i'', b''), E_i'')$$

(see (cf. [EV2, 9.5], particularly (9.5.7)), with the following properties:

- (a) $\underline{\text{Max}} t_i^d = \text{Sing}(J_i'', b'')$.
- (b) Any sequence of transformations at permissible centers $C_j \subset \text{Sing}(J_j'', b'')$,

$$(5.14.1) \quad (W_i, (J_i'', b''), E_i'') \longleftarrow \cdots \longleftarrow (W_s, (J_s'', b''), E_s'')$$

induces a sequence with same centers $C_j \subset \text{Sing}(J_j, b)$,

$$(5.14.2) \quad (W_i, (J_i, b), E_i) \longleftarrow \cdots \longleftarrow (W_s, (J_s, b), E_s)$$

and has the following properties:

- (i) If $\max t_i^d = \cdots = \max t_{s-1}^d$ then $\underline{\text{Max}} t_j^d = \text{Sing}(J_j'', b'')$ for $i \leq j < s$.
- (ii) $\text{Sing}(J_i'', b'') = \emptyset$ if and only $\max t_{s-1}^d > \max t_s^d$.

Step 2. Let $R(1)(\underline{\text{Max}} t_i^d)$ be the union of components of $\underline{\text{Max}} t_i^d$ which have codimension 1 in W_i .

- (1) If $R(1)(\underline{\text{Max}} t_i^d) \neq \emptyset$ then $R(1)(\underline{\text{Max}} t_i^d)$ is regular, and it is open and closed in $\text{Sing}(J'', b'')$. Furthermore, it has normal crossings with E . In this case this is our canonical choice of center and our resolution function f_i^d will be defined so that $\underline{\text{Max}} f_i^d = R(1)(\underline{\text{Max}} t_i^d)$. Therefore we blow-up at $R(1)(\underline{\text{Max}} t_i^d)$,

$$(5.14.3) \quad (W_i, (J_i, b), E_i) \longleftarrow (W_{i+1}, (J_{i+1}, b), E_{i+1}).$$

and either $\max t_i^d > \max t_{i+1}^d$ or $R(1)(\underline{\text{Max}} t_{i+1}^d) = \emptyset$.

- (2) If $R(1)(\underline{\text{Max}} t_i^d) = \emptyset$ then we associate to $(W_i, (J_i'', b''), E_i'')$ a $d - 1$ -dimensional basic object,

$$(5.14.4) \quad (Z_i, (C(J_i''), b''!), \overline{E}_i''),$$

where $Z_i \subset W_i$ is a smooth subscheme of dimension $d - 1$, \overline{E}_i'' is the restriction of E_i'' to Z_i , and $C(J_i'')$ is the *coefficient ideal* of J'' (cf. [EV2, 9.3]).

This $d - 1$ dimensional basic object has the following properties:

- (a) $\underline{\text{Max}} t_0^i = \text{Sing}(C(J_i''), b''!)$.
 (b) Any sequence of transformations

$$(5.14.5) \quad (Z_i, (C(J_i''), b''!), \overline{E}_i) \longleftarrow \cdots \longleftarrow (Z_s, (C(J_s''), b''!), \overline{E}_s)$$

at permissible centers $D_j \subset \text{Sing}((C(J_j''), b''!))$ induces a sequence with same centers $D_j \subset \text{Sing}(J_j, b)$,

$$(5.14.6) \quad (W_i, (J_i, b), E_i) \longleftarrow \cdots \longleftarrow (W_s, (J_s, b), E_s)$$

and

- (i) If $\max t_i^d = \cdots = \max t_{s-1}^d$ then $\underline{\text{Max}} t_j^d = \text{Sing}(C(J_j''), b''!)$ for $i \leq j < s$.
 (ii) $\text{Sing}(C(J_i''), b''!) = \emptyset$ if and only $\max t_{s-1}^d > \max t_s^d$.

Step 3. If $\max w\text{-ord}_i^j > 0$ then Apply the whole process from Step 1, now to the $d - 1$ -dimensional basic object $(Z_i, (C(J_i''), b''!)) = (Z_i, (J_i^{d-1}, b''!))$ so as to define

$$t_i^{d-1} : \text{Sing}(C(J_i''), b''!) \longrightarrow (\mathbb{Q} \times \mathbb{Z}, \leq).$$

The successive application of steps 1, 2 and 3 leads us to define functions

$$t_i^j : \text{Sing}(J_i^j, b) \longrightarrow (\mathbb{Q} \times \mathbb{Z}, \leq),$$

for $j = d, d - 1, \dots, 1$. However note that the functions

$$t_i^j : \text{Sing}(J_i, b) \longrightarrow (\mathbb{Q} \times \mathbb{Z}, \leq)$$

are only defined if $\max w\text{-ord}_i^j > 0$ (see Definition 5.10). If $\max w\text{-ord}_i^j = 0$ then a resolution of the corresponding basic object $(W_i, (J_i^j, b), E_i)$ is simple, and can also be achieved by blowing up at the maximum of a suitable defined function which values in a totally ordered set $(T, <)$ containing $(\mathbb{Q} \times \mathbb{Z}, \leq)$, but this does not require a form of induction on the dimension. If we still denote such function by t_i^d , say

$$t_i^j : \text{Sing}(J_i^j, b) \longrightarrow (T, \leq),$$

then the successive application of Steps 1, 2 and 3 permits to define a function:

$$f_i^d : \text{Sing}(J_0, b) \longrightarrow \overbrace{T \times T \times \cdots \times T}^{k\text{-times}}, \quad 1 \leq k \leq d$$

which is upper semi-continuous and verifies the conditions required in Definition 5.2. We refer here to in [EV2, 7.11] for a precise description of this invariant.

Remark 5.15. A major property of the functions t_i^d is that these locally defined functions t_0^{d-1} patch so as to define a global function t_i^{d-1} , and the locally defined resolutions (5.14.1), (5.14.2), (5.14.5) and (5.14.6) are sufficiently natural so as to define a (global) resolution of the basic object $(W_i, (J_i, b)E_i)$. This is discussed in [EV2, 9.5].

Remark 5.16. The proof of Lemma 4.8 is based on the notion of relative codimension stated in Definitions 4.3 and 4.7. As we mentioned in Remark 5.13 this notion is related to certain values of the functions t_i^d : Let $X \subset W$ be under the assumptions of Theorem 1.2. Then X is regular in $X \setminus \text{Sing}(X)$, which is a dense open set of X . Consider the basic object

$$(W_0, (J_0, 1), E_0) = (W, (\mathcal{I}(X), 1), \emptyset).$$

Note that the order of J_0 is one at every point $y \in X \setminus \text{Sing}(X)$, and in fact for every $y \in X \setminus \text{Sing}(X)$,

$$t_0^d(y) = (1, 0).$$

Note also that J_0 is of local codimension ≥ 1 , for every $y \in X \setminus \text{Sing}(X)$. Consider the constructive resolution of the basic object $(W_0, (J_0, 1), E_0)$,

$$(5.16.1) \quad (W_0, (J_0, 1), E_0) \longleftarrow \dots \longleftarrow (W_N, (J_N, 1), E_N).$$

Since $\text{Sing}(J_N, 1) = \emptyset$, the ideal $J_0 \mathcal{O}_{W_N}$ defines a principalization of J_0 . Now, by property **E** of Definition 5.2 and since $\text{w-ord}_0^d(y) = 1$ for $y \in X \setminus \text{Sing}(X)$, it follows that there must be an index $i_0 \in \{0, 1, \dots, N\}$ in the sequence (5.16.1) such that

$$\max \text{w-ord}_{i_0-1} > \max \text{w-ord}_{i_0} = 1.$$

For such index i_0 ,

$$J_0 \mathcal{O}_{W_{i_0}} = \mathcal{L}_{i_0} \bar{J}_{i_0}$$

and the order of \bar{J}_{i_0} at every point $y \in V(\bar{J}_{i_0})$ is one. Since the sequence (5.16.1) is obtained by blowing-up at t_i^d -permissible centers, we may also choose $i_1 \geq i_0$ so that

$$\max t_{i_1-1}^d > \max t_{i_1}^d = (1, 0).$$

For such index i_1

$$J \mathcal{O}_{W_{i_1}} = \mathcal{L}_{i_1} \bar{J}_{i_1}.$$

We claim now that \bar{J}_{i_1} has (W_{i_1}, E_{i_1}) -codimension ≥ 1 . To see this we fix a point $y_{i_1} \in V(\bar{J}_{i_1})$. We will show that the local condition of Definition 4.3 holds at $\mathcal{O}_{W_{i_1}, y_{i_1}}$. Let y_{i_0} denote the image of y_{i_1} at W_{i_0} . By Remark 5.8

$$\text{w-ord}_{i_0}^d(y_{i_0}) \geq \text{w-ord}_{i_1}^d(y_{i_1}).$$

Since

$$1 = \max \text{w-ord}_{i_0}^d \geq \text{w-ord}_{i_0}^d(y_{i_0}) \geq \text{w-ord}_{i_1}^d(y_{i_1}) \geq 1,$$

it follows that

$$\text{w-ord}_{i_0}^d(y_{i_0}) = \text{w-ord}_{i_1}^d(y_{i_1}) = 1.$$

Since the order of \overline{J}_{i_0} is one, we can choose a regular system of parameters

$$\{x_1, \dots, x_n\} \subset \mathcal{O}_{W_{i_0}, y_{i_0}}$$

so that $x_1 \in \overline{J}_{i_0}$. We still do not know if condition (ii) of (1) in Definition 4.3 holds, since the hypersurfaces $H_j \in E_{i_0}$ may not be transversal to $V(< x_1 >)$. Locally at y_{i_0} set $Z_{i_0} = V(x_1)$; so Z_{i_0} is a smooth hypersurface which contains $V(\overline{J}_{i_0})$. Let C_{i_0} denote the center of the transformation

$$W_{i_0} \longleftarrow W_{i_0+1}.$$

If $y_{i_0} \in C_{i_0}$ then

$$C_{i_0} \subset \underline{\text{Max}} t_{i_0}^d \subset V(\overline{J}_{i_0}) \subset Z_{i_0}$$

and hence, if Z_{i_0+1} denotes the strict transform of Z_{i_0} in W_{i_0+1} , the exceptional divisor intersects Z_{i_0+1} transversally. On the other hand, the law of transformation defining

$$(W_{i_0}, (J_{i_0}, 1), E_{i_0}) \longleftarrow (W_{i_0+1}, (J_{i_0+1}, 1), E_{i_0+1})$$

is such that $\mathcal{I}(Z_{i_0+1}) \subset \overline{J}_{i_0+1}$. In particular the smooth hypersurface $\mathcal{I}(Z_{i_0+1})$ is in the same setting as Z_{i_0} , namely,

$$V(\overline{J}_{i_0+1}) \subset Z_{i_0+1}.$$

Since $i_0 \leq i_1$, we repeat this argument $(i_0 - i_1)$ -times,

$$(5.16.2) \quad (W_{i_0}, E_{i_0}) \longleftarrow (W_{i_0+1}, E_{i_0+1}) \longleftarrow \dots \longleftarrow (W_{i_1}, E_{i_1})$$

and at the end we get that for each index $i_0 \leq j \leq i_1 - 1$

$$(5.16.3) \quad C_j \subset Z_j,$$

where C_j denotes the center of the monoidal transformation $W_j \longleftarrow W_{j+1}$ and $Z_j \subset W_j$ denotes the strict transform of $Z_{i_0} \subset W_{i_0}$. Also

$$V(\overline{J}_l) \subset Z_l,$$

for each index $i_0 \leq l \leq i_1$. Now $y_{i_1} \in V(\overline{J}_{i_1}) \subset Z_{i_1}$, and

$$t_{i_1}^d(y_{i_1}) = (\text{w-ord}_{i_1}^d(y_{i_1}), n_{i_1}^d(y_{i_1})) = (1, 0).$$

Thus condition $n_{i_1}^d(y_{i_1}) = 0$ asserts that none of the strict transforms of hypersurfaces $H_j \in E_{i_0}$ contains the point y_{i_1} , while all the exceptional divisors in sequence (5.16.2) have normal crossings with Z_{i_1} by condition (5.16.3). Therefore \overline{J}_{i_1} has (W_{i_1}, E_{i_1}) -codimension ≥ 1 at y_{i_1} .

Remark 5.17. Case $\max t^d = (1, 0)$. With the same notation as in Remark 5.16, set i_1 so that

$$\max t_{i_1-1}^d > \max t_{i_1}^d = (1, 0).$$

Since $\max \text{w-ord}_{i_1}^d = 1$, it follows that $\underline{\text{Max}} \text{w-ord}_{i_1}^d = V(\overline{J}_{i_1})$. Furthermore, since $t_{i_1}^d$ is an upper semi-continuous function, it also follows that $\underline{\text{Max}} t_{i_1}^d = V(\overline{J}_{i_1})$. If $\max t_{i_1}^d = (1, 0)$ then the basic object $(W_{i_1}, (J''_{i_1}, b''), E''_{i_1})$ attached to $\underline{\text{Max}} t_{i_1}^d$ as in (2) of Step 2 in 5.14 is defined by setting $b'' = 1$, $J''_{i_1} = \overline{J}_{i_1}$ and $E''_{i_1} = E_{i_1}$ (see [EV2, 9.5]).

Write $J_{i_1} = \mathcal{L}_{i_1} \overline{J}_{i_1}$. Now if $y_{i_1} \in V(\overline{J}_{i_1})$ then there is a regular system of parameters $\{x_1, \dots, x_n\} \subset \mathcal{O}_{W_{i_1}, y_{i_1}}$ such that

- (i) $\langle x_1 \rangle \subset (\overline{J}_{i_1})_{y_{i_1}}$.
- (ii) $(\mathcal{L}_{i_1})_{y_{i_1}} = x_{j_1}^{c_1} \cdot \dots \cdot x_{j_r}^{c_r}$, with $1 < j_1 < \dots < j_r < n$.

If in addition $R(1)(\underline{\text{Max}} t_{i_1}^d) = \emptyset$ locally at y_{i_1} then the basic object $(Z_{i_1}, (C(J_{i_1}'', b''), \overline{E}_{i_1}''))$ can be defined by setting $Z_{i_1} = V(\langle x_1 \rangle)$, $C(J_{i_1}'')$ as the trace of \overline{J}_{i_1} at $\mathcal{O}_{W_{i_1}, y_{i_1}} / \langle x_1 \rangle$ and \overline{E}_{i_1}'' as the restriction of E_{i_1}'' to Z_{i_1} (see [EV2, 9.3]).

6. EXAMPLES

The purpose of this section is to illustrate with some examples how the functions f_i^d and t_i^d are defined.

Example 6.1. Let us consider again $W = \mathbb{A}_{\mathbb{Q}}^3 = \text{Spec } \mathbb{Q}[x_1, x_2, x_3]$, and let X be the irreducible curve determined by the ideal $J = \langle x_1, x_2x_3 + x_2^3 + x_3^3 \rangle = \mathcal{I}(X)$.

Consider the basic object $(W, (J, 1), E = \{\emptyset\}) = (W_0, (J_0, 1), E_0)$ and set $f_0 = x_2x_3 + x_2^3 + x_3^3$. Note that $\text{Sing}(J_0, 1) = X$ and that the order of the ideal J_0 at every point is 1. Since $E_0 = \{\emptyset\}$ we see that $t_0^3(\xi) = (1, 0)$ for any $\xi \in \text{Sing}(J, 1)$, and hence $\max t_0^3 = (1, 0)$ and $\underline{\text{Max}} t_0^3 = \text{Sing}(J, 1) = V(J)$.

Now we attach to $\underline{\text{Max}} t_0^3$ a basic object $(W_0, (J_0'', b''), E_0'')$ in the sense of step 1 of (5.14). In this case by 5.17, $J_0'' = J_0$, $b'' = 1$ and $E_0'' = \{\emptyset\}$.

Note that $R(1)(\underline{\text{Max}} t_0^3) = R(1)(\text{Sing}(J, 1)) = \emptyset$ since $\text{Sing}(J, 1)$ is a curve in a 3 space, so we proceed by induction in the sense of Step 2 of 5.14. Following 5.17 we choose the hypersurface $Z \subset W$ determined by the ideal $\mathcal{I}(Z) = \langle x_1 \rangle$, and define

$$(6.1.1) \quad C(J'') = \langle x_2x_3 + x_2^3 + x_3^3 \rangle \subset \mathbb{Q}[x_2, x_3] \simeq \mathbb{Q}[x_1, x_2, x_3] / \langle x_1 \rangle,$$

and $b''! = 1$. So $Z = \mathbb{A}_{\mathbb{Q}}^2 \subset \mathbb{A}_{\mathbb{Q}}^3 = W$.

Recall that a resolution of the basic object $(W_0, (J_0'', b''), E_0)$ is equivalent to a resolution of $(Z, (C(J''), b''!), E_Z)$.

Now we define the function $t_0^2 : \text{Sing}(C(J''), 1) \longrightarrow \mathbb{Q} \times \mathbb{Z}$ by

$$(6.1.2) \quad t_0^2(\eta) = \begin{cases} (2, 0) & \text{if } \eta = (\mathbf{0}, \mathbf{0}) \in \mathbb{A}_{\mathbb{Q}}^2 \\ (1, 0) & \text{if } \eta \in X \setminus \{(\mathbf{0}, \mathbf{0})\} \end{cases}$$

Hence $\underline{\text{Max}} t_0^2 = (\mathbf{0}, \mathbf{0})$. Since $\underline{\text{Max}} t_0^2$ is a closed point and the resolution function described in 5.14 is defined in terms of the functions t_i^j , it follows that we blow-up at this closed point,

$$(6.1.3) \quad (Z_0, (\langle f_0 \rangle, 1), (E_Z)_0) \longleftarrow (Z_1, (\langle f_1 \rangle, 1), (E_Z)_1 = \{\overline{H}_1\}),$$

where \overline{H}_1 is the exceptional divisor and f_1 is the strict transform of f_0 . Note that

$$(6.1.4) \quad < f_0 > = \mathcal{I}(\overline{H}_1) \cdot (< f_1 >).$$

Sequence (6.1.3) induces a sequence

$$(6.1.5) \quad (W_0, (J_0, 1), E = \{\emptyset\}) \longleftarrow (W_1, (J_1, 1), E_1 = \{H_1\}),$$

and in this case $J_1 = \overline{J}_1$ (in the sense of (3.3.4)).

Let $X_1 \subset W_1$ be the strict transform of $X \subset W$, and let $\{p, q\} = X_1 \cap H_1$. Now we will describe locally $J_0 \mathcal{O}_{W_1, p}$: There is a regular system of parameters $\{\frac{x_1}{x_2}, x_2\}$ at $\mathcal{O}_{W_1, p}$, such that $J_0 \mathcal{O}_{W_1, p} = < \frac{x_1}{x_2}, x_2(f_1) >$, and hence

$$(6.1.6) \quad J_0 \mathcal{O}_{W_1, p} \subset < \frac{x_1}{x_2}, x_2 >.$$

Let $L_1 = V(< \frac{x_1}{x_2}, x_2 >)$. Then $L_1 \subset H_1$ and $\mathcal{I}(L_1)$ will be an embedded component of the total transform of J_0 at \mathcal{O}_{W_1} . In particular this embedded component will arise after any sequence of quadratic transformations, and hence Hironaka's desingularization does not fulfill property (iii) of Theorem 1.2.

Following the resolution algorithm proposed in [EV3] (see also the addendum of [EV2]), the desingularization comes to an end after blowing up at the closed points $\{p, q\}$, and therefore the same argument as before shows that property (iii) of Theorem 1.2 does not hold for that algorithm.

Example 6.2. Now consider $W = \mathbb{A}_{\mathbb{Q}}^4 = \text{Spec } \mathbb{Q}[x_0, x_1, x_2, x_3]$ and let X be the irreducible closed subscheme determined by the ideal $J = < x_0, x_1, x_2 x_3 + x_2^3 + x_3^3 > = \mathcal{I}(X)$. Let $(W, (J, 1), E = \{\emptyset\}) = (W_0, (J_0, 1), E_0)$ be the corresponding basic object. Set $f_0 = x_2 x_3 + x_2^3 + x_3^3$. Once more the function t_0^4 is constant on $\text{Sing}(J_0, 1)$ with value $(1, 0)$, and hence $\max t_0^4 = (1, 0)$ and $\underline{\text{Max}} t_0^4 = \text{Sing}(J_0, 1) = X$.

As in Step 1 of 5.14, we associate to $\underline{\text{Max}} t_0^4 = \text{Sing}(J_0, 1)$ the basic object $(W, (J'', b''), E)$, where now $J'' = J$, $b'' = 1$ and $E = \{\emptyset\}$ (see 5.17). So a resolution of this basic object is a resolution of $(W_0, (J_0, b), E_0)$.

Note that $R(1)(\text{Sing}(J'', b'')) = \emptyset$, so we proceed by induction as in Step 2 of 5.14. Let Z be the regular hypersurface determined by the ideal $\mathcal{I}(Z) = < x_0 >$, let

$$(6.2.1) \quad C(J'') = < x_1, f > \subset \mathbb{Q}[x_1, x_2, x_3] \simeq \mathbb{Q}[x_0, x_1, x_2, x_3] / < x_0 >$$

and $b''! = 1$. Then a resolution of the basic object $(Z, (C(J''), b''!), (E_Z))$ is equivalent to a resolution of the basic object $(W, (J'', b''), E)$.

Note that the function t_0^3 is constant along $\text{Sing}(C(J''), 1)$ and its value is $(1, 0)$, hence

$$\underline{\text{Max}} t_0^3 = \text{Sing}(C(J''), 1).$$

The next step of the algorithm of resolution is to associate a basic object,

$$(Z'', (C(J'')'', e), E_Z),$$

to $(Z, (C(J''), 1), E_Z)$ in the sense of Step 1 of 5.14. Note that at this point we are in the same situation as in Example 6.1, thus a similar argument will tell us that we will define a function t_0^2 in a two dimensional ambient space with constant value equal to $(1, 0)$.

Summarizing we have constructed the first three coordinates of the function of the algorithm of resolution at the first stage:

$$(6.2.2) \quad f_0^4 : X_0 \longrightarrow (\mathbb{Q} \times \mathbb{Z}) \times (\mathbb{Q} \times \mathbb{Z}) \times (\mathbb{Q} \times \mathbb{Z})$$

where,

$$\xi \longrightarrow f_0^4(\xi) = \begin{cases} [(1, 0), (1, 0), (2, 0)] & \text{if } \xi = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \in \mathbb{A}_{\mathbb{Q}}^4 \\ [(1, 0), (1, 0), (1, 0)] & \text{if } \xi \in X \setminus (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \end{cases}$$

Example 6.3. *Canonical choice of the centers.* Let $W = \mathbb{A}_{\mathbb{Q}}^3 = \text{Spec } \mathbb{Q}[x_1, x_2, x_3]$,

$$(6.3.1) \quad J = \langle x_1, x_2 \rangle$$

and consider the associated basic object $(W, (J, 1), E = \{\emptyset\})$. The function t_0^3 is constant on $V(J)$ with value $(1, 0)$ and hence $\underline{\text{Max}} t_0^3 = \text{Sing}(J, 1)$.

We associate the basic object $(W, (J'', b''), E = \{\emptyset\})$ to $\underline{\text{Max}} t_0^3$ in the sense of Step 1 of 5.14, where, once more, $J'' = J$ and $b'' = 1$.

As $R(1)(\text{Sing}(J'', b'')) = \emptyset$, we are under the assumptions Step 2 of 5.14, and hence we define $Z = V(\langle x_1 \rangle)$, $C(J'') = \langle x_2 \rangle \subset \mathbb{Q}[x_2, x_3] \simeq \mathbb{Q}[x_1, x_2, x_3] / \langle x_1 \rangle$, and $b'' = 1$.

Now we consider the basic object $(Z, (C(J''), 1), (E_Z)_0)$. Note that

$$\underline{\text{Max}} t_0^2 = R(1)(\underline{\text{Max}} t_0^3) = \text{Sing}(C(J''), 1) = V(x_1, x_2) \subset \mathbb{A}_{\mathbb{Q}}^3,$$

and this is the canonical choice of center mentioned in (1) of step 2 in 5.14.

Remark 6.4. Let us assume that $J\mathcal{O}_W$ is an ideal defining a closed regular and pure dimensional scheme X of codimension e and consider the basic object $(W, (J, 1), E = \{\emptyset\})$. Then $\max t^d = (1, 0)$, $\underline{\text{Max}} t^d = X$ and $(J'', b'') = (J, 1)$ as in Remark 5.17. Locally at any point $\xi \in X$, the basic object $(Z, (C(J''), l), E_Z)$ is equal to $(Z, (J|_Z, 1), \emptyset)$. Now we are dealing with a $d - 1$ -dimensional basic object where $V(J|_Z)$ determines a regular scheme of codimension equal to the codimension of X in W (see Remark 5.17). So $(Z, (C(J''), l), \emptyset)$ is under the

same conditions as $(W, (J, 1), E = \emptyset)$. Therefore the first e coordinates of the function f^d that defines the algorithm are:

$$(6.4.1) \quad \begin{array}{ccc} f^{d,e} : & X & \longrightarrow \overbrace{(\mathbb{Q} \times \mathbb{Z}) \times \dots \times (\mathbb{Q} \times \mathbb{Z})}^{e\text{-times}} \\ & \xi & \longrightarrow (1, 0) \times \dots \times (1, 0) \end{array}$$

7. PROOF OF LEMMA 4.8

Definition 7.1. Let $X \subset W$ be under the assumptions of Theorem 1.2, let $W_0 = W$, and $J_0 = J = \mathcal{I}(X)$. Consider the basic object $(W_0, (J_0, 1), E_0)$ (note that in this case $\text{Sing}(J_0, 1) = X$). With this notation define

$$\text{RSing}(X) = \{y \in \text{Sing}(J_0, 1) : y \in H_i \text{ for some } E_i \in E_0, \text{ or } J_0 \text{ is not regular at } y\}$$

The set $\text{RSing}(X)$ will be called the *relative singular locus* since the notion is relative to E . Note that $\text{RSing}(X) \subset X$ is closed. If $E_0 = \emptyset$ then $\text{RSing}(X) = \text{Sing}(X)$. Although in the formulation of Theorem 1.2 we assume $E_0 = \emptyset$, in our proofs, based in inductive arguments and step by step procedures, we will have to consider the case when $E_0 \neq \emptyset$.

Definition 7.2. [*The Relative Property.*] With the same notation and assumptions as in Definition 7.1, consider a sequence of monoidal transformations,

$$(7.2.1) \quad (W_0, (J_0, 1), E_0) \longleftarrow \dots \longleftarrow (W_N, (J_N, 1), E_N)$$

at permissible centers $C_i \subset W_i$, and let $F \subset W_0$ be the union of all images of C_i (which will be a closed subset of W_0 since all morphisms $W_i \rightarrow W_0$ are proper). We say that the sequence (7.2.1) has the *relative property*, if

$$(7.2.2) \quad F \subset \text{RSing}(X).$$

Definition 7.3. Given a sheaf of ideals $J \subset \mathcal{O}_W$, we will denote by $R(a)(J)$ the union of components of $V(J)$ of codimension a . Note that $R(a)(J)$ may be empty.

Proof of Lemma 4.8

Let $(W_0, (J_0, 1), E_0)$ be as in Theorem 1.2. Note that every sequence of transformations of pairs

$$(W_0, E_0) \longleftarrow \dots \longleftarrow (W_N, E_N)$$

induces a factorization $J_0 \mathcal{O}_{W_N} = \mathcal{L}_N \bar{J}_N$. We will prove Lemma 4.8 by induction on the relative codimension of the ideal \bar{J}_N . The two steps of the inductive argument are presented in 7.4 and 7.6.

7.4. Case 0. Let $X \subset W$ be under the assumptions of Theorem 1.2, and let $(W_0, (J_0, 1), E_0) = (W, (\mathcal{I}(X), 1), \emptyset)$. Then there is a finite sequence of transformations of basic objects

$$(7.4.1) \quad (W_0, (J_0, 1), E_0) \longleftarrow \dots \longleftarrow (W_{r_1}, (J_{r_1}, 1), E_{r_1})$$

so that $J_{r_1} \mathcal{O}_{W_{r_1}} = \bar{J}_{r_1}$, and \bar{J}_{r_1} is of (W_{r_1}, E_{r_1}) -codimension ≥ 1 .

The sequence (7.4.1) will be defined as a concatenation of two chains of transformations: sequences (7.4.2) and (7.4.5), which will be constructed, respectively, in two steps, **A** and **B**. The condition of relative codimension ≥ 1 will be achieved really in Step **A**, but Step **B** will play a role in our inductive argument.

Step A.

There is a sequence of transformations at permissible centers $C_i \subset \text{Sing}(J_1, 1)$,

$$(7.4.2) \quad (W_0, (J_0, 1), E_0) \longleftarrow \cdots \longleftarrow (W_{k_1}, (J_{k_1}, 1), E_{k_1})$$

so that

$$\max t_0^d \geq \max t_1^d \geq \max t_2^d \cdots \geq \max t_{k_1-1}^d > \max t_{k_1}^d$$

and $\max t_{k_1}^d = (1, 0)$ but $\max t_i^d > (1, 0)$ for any index $i < k_1$. If $\max t_0^d = (1, 0)$ we would take $k_1 = 0$. Then,

$$(7.4.3) \quad J_{k_1} = I(H_1)^{a_1} \cdots I(H_{k_1})^{a_{k_1}} \overline{J}_{k_1}$$

where $a_1, \dots, a_{k_1} \in \mathbb{N}$, and by Remark 5.16, \overline{J}_{k_1} is of (W_{k_1}, E_{k_1}) -codimension ≥ 1 .

Let $U_0 = W_0 - \text{RSing}(X)$. Note that U_0 is dense in W_0 , and that $X \cap U_0$ is a smooth subscheme of U_0 . Define

$$U_{k_1} \subset W_{k_1}$$

as the pull-back of U_0 in W_{k_1} , let F_{k_1} be the union of the images of C_i in W_0 , let

$$V_0 = W_0 \setminus F_{k_1}$$

and set

$$V_{k_1} = W_{k_1} \setminus \bigcup_{H_i \in E_{k_1}} H_i.$$

Step B.

In expression (7.4.3) consider the smallest index j so that $a_j \geq 1$. Since $a_j \geq 1$, then $H_j \subset \text{Sing}(J_{k_1}, 1)$, and it is also clear that H_j has normal crossings with E_{k_1} . Blow-up at H_j :

$$(W_{k_1}, E_{k_1}) \longleftarrow (W_{k_1+1}, E_{k_1+1}).$$

Then $W_{k_1} = W_{k_1+1}$, $E_{k_1+1} = E_{k_1}$ and

$$(7.4.4) \quad J_{k_1+1} = I(H_1)^{a_1^*} \cdots I(H_j)^{a_j^*} \overline{J}_{k_1}$$

where $a_i^* = a_i$ for each $i \in \{1, \dots, k_1\}$, $i \neq j$, and $a_j^* = a_j - 1$ (see (3.3.4) for the way that J_{k_1+1} is defined).

In this way we define, for some index $r_1 \geq k_1$, a sequence of monoidal transformations :

$$(7.4.5) \quad (W_{k_1}, (J_{k_1}, 1), E_{k_1}) \longleftarrow \cdots \longleftarrow (W_{r_1}, (J_{r_1}, 1), E_{r_1}),$$

so that

$$(7.4.6) \quad (W_{k_1}, E_{k_1}) = (W_{r_1}, E_{r_1})$$

but now expression (7.4.3) reduces to

$$(7.4.7) \quad J_{r_1} = \overline{J}_{r_1}$$

Note that $W_{r_1} \rightarrow W_{k_1}$ is the identity map. Set V_{r_1} and U_{r_1} as pull-backs of V_{k_1} and U_{k_1} at W_{r_1} .

Remark 7.5.

- i. Note that $\max t_{r_1}^d = (1, 0)$, thus $\underline{\text{Max}} t_{r_1}^d = V(\overline{J}_{r_1})$ (so $t_{r_1}^d(y) = (1, 0)$ for every $y \in V(\overline{J}_{r_1})$), and therefore by Remark 5.16 \overline{J}_{r_1} is of (W_{r_1}, E'_{r_1}) -codimension ≥ 1 .
- ii. The restriction of $\overline{J}_{r_1}(\subset \mathcal{O}_{W_{r_1}})$ to V_{r_1} (respectively to U_{r_1}) is naturally identified with J_0 restricted to V_0 (respectively with J_0 restricted to U_0).
- iii. Since Definition 4.3 holds for \overline{J}_{r_1} and (W_{r_1}, E_{r_1}) with $a = 1$, at any closed point $y \in V(\overline{J}_{r_1})$, there is a regular system of parameters $\{x_1, x_2, \dots, x_d\} \subset \mathcal{O}_{W_{r_1}, y}$ such that

$$\langle x_1 \rangle \subset (\overline{J}_{r_1})_y$$

and if $y \in H_i \in E_{r_1}$ then $\mathcal{I}(H_i) = \langle x_{i_j} \rangle$ with $i_j > 1$. This leads to:

- iv. If X is a subscheme of codimension > 1 in W_0 , then $R(1)(V(\overline{J}_{r_1}))$ must be empty. In fact, note that by (iii), $R(1)(V(\overline{J}_{r_1})) \cap V_{r_1}$ must be dense in $R(1)(V(\overline{J}_{r_1}))$; but V_{r_1} is isomorphic to an open subset of W_0 . The claim follows now from (ii), and the assumption on the codimension of X .

Claim: *The concatenation of sequences (7.4.2) and (7.4.5):*

$$(7.5.1) \quad (W_0, (J_0, 1), E_0) \longleftarrow \cdots \longleftarrow (W_{k_1}, (J_{k_1}, 1), E_{k_1}) \longleftarrow \cdots \longleftarrow (W_{r_1}, (J_{r_1}, 1), E_{r_1})$$

has the relative property introduced in 7.2.

Proof of the Claim: It suffices to check, that for each index $i = 1, \dots, r_1$, the proper morphism $W_i \rightarrow W_0$ induces an isomorphism over $U_0 = W_0 - \text{RSing}(X)$.

For the construction in Step A, we may assume by induction that $W_{i-1} \rightarrow W_0$ induces an isomorphism over $U_0 = W_0 - \text{RSing}(X)$, and note that $C_{i-1} \subset \underline{\text{Max}} t_{i-1}^d$. Since $\max t_{i-1}^d > (1, 0)$ it suffices to recall that $t_0^d(y) = (1, 0)$ for any point $y \in \text{Sing}(J_0, 1) \cap U_0$.

The same statement of the claim is clear from the construction in Step B, since only exceptional hypersurfaces are chosen as centers. This proves the claim.

7.6. Case $e \geq 1$. *Assume that X is of codimension $> e$ in W_0 , and that there is a finite sequence of transformations of basic objects*

$$(7.6.1) \quad (W_0, (J_0, 1), E_0) \longleftarrow \cdots \longleftarrow (W_{k_e}, (J_{k_e}, 1), E_{k_e})$$

at permissible centers C_i so that:

- (1) If F_{r_e} is the union of the images of C_i in W_0 , if $V_0^e = W_0 \setminus F_{r_e}$, and if $V_{r_e} \subset W_{r_e}$ is the pull-back of V_0^e in W_{r_e} , then

$$W_0 \longleftarrow W_{r_e}$$

defines an isomorphism $V_0^e \cong V_{r_e}$ and $F^e \subset \text{RSing}(X)$ (i.e the relative property in Definition 7.2 holds); and hence $U_0 \subset V_0^e$. In particular, if $U_{r_e} \subset W_{r_e}$ denotes the pull-back of U_0 , then $U_{r_e} \subset V_{r_e}$ and $U_{r_e} \cong U_0$.

- (2) \bar{J}_{r_e} has relative codimension $\geq e$.

- (3) $R(e)(V(\bar{J}_{r_e})) = \emptyset$.

Then under this assumptions there is an enlargement of the sequence (7.6.1),

$$(7.6.2) \quad (W_0, (J_0, 1), E_0) \longleftarrow \cdots \longleftarrow (W_{r_e}, (J_{r_e}, 1), E_{r_e}) \longleftarrow \cdots \longleftarrow (W_{r_{e+1}}, (J_{r_{e+1}}, 1), E_{r_{e+1}})$$

so that $J_{r_{e+1}} = \bar{J}_{r_{e+1}}$ and $\bar{J}_{r_{e+1}}$ has relative codimension $\geq e + 1$.

We will accomplish this part of the proof in two steps, **A** and **B**.

Step A.

Under the assumptions of 7.6, we may assume that locally at $y \in V(J_{r_e})$:

- (i) All functions $t_{r_e}^d, t_{r_e}^{d-1}, \dots, t_{r_e}^{d-e-1}$ are defined,
- (ii) $\max t_{r_e}^d = (1, 0), \max t_{r_e}^{d-1} = (1, 0), \dots, \max t_{r_e}^{d-e-1} = (1, 0)$
- (iii) By (2) there is a regular system of parameters $\{x_1, \dots, x_e\} \subset \mathcal{O}_{W_{r_e}, y}$, so that:
 - (a) $\langle x_1, x_2, \dots, x_e \rangle \subset (\bar{J}_{r_e})_y \subset \mathcal{O}_{W_{r_e}, y}$.
 - (b) For each $H_i \in E_{r_e}$ with $y \in H_i$ there exists $j_i > e$ so that

$$\mathcal{I}(H_i)_y = \langle x_{j_i} \rangle.$$

- (iv) By (3), $\langle x_1, x_2, \dots, x_e \rangle \neq (\bar{J}_{r_e})_y$.

- (v) Since by (3) $R(e)(V(\bar{J}_{r_e})) = \emptyset$, by Remark 5.17,

$$(7.6.3) \quad (V(x_1, \dots, x_e), (\bar{J}_{r_e}|_{V(\langle x_1, \dots, x_e \rangle)}, 1), (E_{r_e})|_{V(\langle x_1, \dots, x_e \rangle)}) =$$

$$(\bar{W}_{r_e}, (\mathcal{A}_{r_e}, 1), \bar{E}_{r_e})$$

is the $d - e$ dimensional basic object attached to $\underline{\text{Max}} t_{k_e}^{d-e}$ in a suitable neighborhood of $y \in W_{r_e}$. Note that $\mathcal{A}_{r_e} \neq 0$. This can be done for an open covering, and these are the locally defined basic objects attached to the value

$$\underbrace{[(1, 0), \dots, (1, 0)]}_{e\text{-times}}$$

which are the first e -coordinates of the function

$$f_{r_e}^d : \text{Sing}(J_{r_e}, 1) \longrightarrow \overbrace{T \times \dots \times T}^{d\text{-times}},$$

(see 5.14).

By assumption we know that $J_{r_e} = \overline{J}_{r_e}$ is an ideal of order 1 at every point $y \in \text{Sing}(J_{r_e}, 1)$. In a suitable open neighborhood of y :

$$(7.6.4) \quad \text{Sing}(J_{r_e}, 1) \subset \overline{W}_{r_e} = V(< x_1, x_2, \dots, x_e >)$$

and $\text{Sing}(J_{r_e}, 1) = \text{Sing}(\mathcal{A}_{r_e}, 1)$ (at least locally).

Note that $\dim(\overline{W}_{r_e}) = (d - e)$, so $t_{r_e}^{d-e}$ is defined as a function on $\text{Sing}(\mathcal{A}_{r_e}, 1)$. By the previous identifications, we can view, at least locally, $t_{r_e}^{d-e}$ as a function on $\text{Sing}(J_{r_e}, 1)$. By Remark 5.15 these locally defined functions $t_{r_e}^{d-e}$ patch so as to define a function on all $V(J_{r_e}) = \text{Sing}(J_{r_e}, 1)$. Furthermore, there is a sequence of transformations of basic objects

$$(7.6.5) \quad (W_{r_e}, (J_{r_e}, 1), E_{r_e}) \longleftarrow \dots \longleftarrow (W_{k_{e+1}}, (J_{k_{e+1}}, 1), E_{k_{e+1}})$$

so that

$$(7.6.6) \quad J_j = \overline{J}_j$$

for each index $j = r_e, r_e + 1, \dots, k_{e+1}$, which locally induces the sequence:

$$(7.6.7) \quad (\overline{W}_{r_e}, (\mathcal{A}_{r_e}, 1), \overline{E}_{r_e}) \longleftarrow \dots \longleftarrow (\overline{W}_{k_{e+1}}, (\mathcal{A}_{k_{e+1}}, 1), \overline{E}_{k_{e+1}})$$

such that

$$\max t_{r_e}^{d-e} \geq \max t_{r_e+1}^{d-e} \geq \dots \geq \max t_{k_{e+1}-1}^{d-e} > \max t_{k_{e+1}}^{d-e} = (1, 0)$$

And hence

$$(7.6.8) \quad \mathcal{A}_{k_{e+1}} = \mathcal{I}(H_{r_e+1})^{a_1} \dots \mathcal{I}(H_{k_{e+1}})^{a_{k_{e+1}-r_e}} \overline{\mathcal{A}}_{k_{e+1}}$$

where now $\overline{\mathcal{A}}_{k_{e+1}}$ has relative codimension ≥ 1 (same argument as in Step A of Case 0, see also part (i) of Remark 7.7).

On the local description of $J_{k_{e+1}}$:

Fix $y \in V(J_{k_{e+1}})$. At $\mathcal{O}_{W_{k_{e+1}}, y}$ there is a regular system of parameters $\{x_1, x_2, \dots, x_d\}$ and there are ideals $M_{k_{e+1}}$ and $\mathcal{D}_{k_{e+1}}$, so that

$$(7.6.9) \quad (J_{k_{e+1}})_y = < x_1, x_2, \dots, x_e > + M_{k_{e+1}} \cdot \mathcal{D}_{k_{e+1}}$$

where:

- i. $\mathcal{D}_{k_{e+1}}$ induces $\overline{\mathcal{A}}_{k_{e+1}}$ (modulo $< x_1, x_2, \dots, x_e >$), and $M_{k_{e+1}}$ is a monomial that induces $I(H_{r_e+1})^{a_1} \dots I(H_{k_{e+1}})^{a_{k_{e+1}-r_e}}$ modulo $< x_1, \dots, x_e >$ (see (7.6.8)).
- ii. $x_{e+1} \in \mathcal{D}_{k_{e+1}}$ and $M_{k_{e+1}}$ is a monomial in coordinates x_s involving only indices $s > e + 1$ (i.e. $x_{e+1} \cdot M_{k_{e+1}} \in (J_{k_{e+1}})_y$).

Step B.

Now we get rid of the monomial $M_{k_{e+1}}$: In expression (7.6.8), consider the smallest index j so that there is a chart for which $a_j \geq 1$, and blow up the hypersurface \overline{H}_j . Since $a_j \geq 1$, then $\overline{H}_j \subset \text{Sing}(\overline{\mathcal{A}}_{k_{e+1}}, 1)$, and it is clear that \overline{H}_j has normal crossings with $\overline{E}_{k_{e+1}}$. In other words, take the smallest index j so that

$$(7.6.10) \quad \dim(H_j \cap V(J_{k_{e+1}})) = d - e - 1$$

Consider the blowing up at H_j ,

$$(W_{k_{e+1}}, E_{k_{e+1}}) \longleftarrow (W_{(k_{e+1})+1}, E_{(k_{e+1})+1})$$

Note that $\overline{W}_{k_{e+1}} = \overline{W}_{(k_{e+1})+1}$, $\overline{E}_{k_{e+1}} = \overline{E}_{(k_{e+1})+1}$ and that

$$(7.6.11) \quad \mathcal{A}_{(k_{e+1})+1} = I(H_{r_{e+1}})^{a_1^*} \cdots I(H_{k_{e+1}})^{a_r^*} \overline{\mathcal{A}}_{(k_{e+1})+1}$$

where $a_i^* = a_i$ for each $i \in \{r_e + 1, \dots, k_{e+1}\}$, $i \neq j$, and $a_j^* = a_j - 1$. Note also that

$$(7.6.12) \quad J_{(k_{e+1})+1} = \overline{J}_{(k_{e+1})+1}$$

and that $\text{Sing}(J_{(k_{e+1})+1}, 1) \subset \overline{W}_{(k_{e+1})+1}$ and $\text{Sing}(J_{(k_{e+1})+1}, 1) = \text{Sing}(\mathcal{A}_{(k_{e+1})+1}, 1)$ (this follows from the properties listed in Step 2 of 5.14).

We repeat this argument over and over so as to define a sequence of transformations of basic objects

$$(7.6.13) \quad (W_{k_{e+1}}, (J_{k_{e+1}}, 1), E_{k_{e+1}}) \longleftarrow \cdots \longleftarrow (W_{r_{e+1}}, (J_{r_{e+1}}, 1), E_{r_{e+1}})$$

which induces a sequence

$$(7.6.14) \quad (\overline{W}_{k_{e+1}}, (\mathcal{A}_{k_{e+1}}, 1), \overline{E}_{k_{e+1}}) \longleftarrow \cdots \longleftarrow (\overline{W}_{r_{e+1}}, (\mathcal{A}_{r_{e+1}}, 1), \overline{E}_{r_{e+1}})$$

such that

$$(7.6.15) \quad \mathcal{A}_{r_{e+1}} = \overline{\mathcal{A}}_{r_{e+1}}.$$

Note that at each index j of the sequence (7.6.13) :

$$(7.6.16) \quad J_j = \overline{J}_j$$

and the local description in the last argument shows that the relative codimension of $J_{r_{e+1}}$ is $\geq e + 1$.

Claim: *The concatenation of the sequences (7.6.5) and (7.6.13):*

$$(7.6.17) \quad (W_0, (J_0, 1), E_0) \longleftarrow \cdots \longleftarrow (W_{r_e}, (J_{r_e}, 1), E_{r_e}) \cdots \longleftarrow (W_{r_{e+1}}, (J_{r_{e+1}}, 1), E_{r_{e+1}})$$

has the relative property introduced in Definition 7.2.

Proof of the Claim: It suffices to check, that for each index i , the induced proper morphism $W_i \rightarrow W_0$ induces an isomorphism over $U_0 = W_0 - \text{RSing}(X)$.

For the construction in Step A, we argue by induction on $i \geq r_e$, so assume that

$$W_{i-1} \rightarrow W_0$$

induces an isomorphism over $U_0 = W_0 - \text{RSing}(X)$, and note that $C_{i-1} \subset \underline{\text{Max}} t_{i-1}^{d-e}$. Since $\max t_{i-1}^{d-e} > (1, 0)$ it suffices to recall that $t_{r_e}^{d-e}(y) = (1, 0)$ for any point $y \in \text{Sing}(J_{r_e}, 1) \cap U_{r_e}$.

The statement of the claim is clear from the construction in Step B, since centers are chosen as exceptional hypersurfaces. This proves the claim.

Remark 7.7. Let $U_{r_{e+1}} \subset W_{r_{e+1}}$ and $V_{r_{e+1}} \subset W_{r_{e+1}}$ denote, respectively, the pull-backs of $U_{r_e} \subset W_{r_e}$ and $V_{r_e} \subset W_{r_e}$ to $W_{r_{e+1}}$. In particular, $U_{r_{e+1}} \subset W_{r_{e+1}}$ denotes the pull-back of U_0 ; there is an inclusion $U_{r_{e+1}} \subset V_{r_{e+1}}$, and $U_{r_{e+1}} \cong U_0$.

- i. Since $\max t_{r_{e+1}}^{d-e} = (1, 0)$ we have that $\underline{\text{Max}} t_{r_{e+1}}^{d-e} = V(\overline{J}_{r_{e+1}})$ (so $t_{r_{e+1}}^{d-e}(y) = (1, 0)$ for every $y \in V(\overline{J}_{r_{e+1}})$). Set $J'' = \overline{J}_{r_{e+1}}$, $b'' = 1$ and $E'' = E_{r_{e+1}}$ (in the sense of 5.14). Then $J'' = \overline{J}_{r_{e+1}}$ is of $(W_{r_{e+1}}, E_{r_{e+1}})$ -relative codimension $\geq e + 1$.
- ii. The restriction of $\overline{J}_{r_{e+1}} \subset \mathcal{O}_{W_{r_{e+1}}}$ to $V_{r_{e+1}}$ (respectively to $U_{r_{e+1}}$) is naturally identified with J_0 restricted to V_0 (respectively with J_0 restricted to U_0).
- iii. Since Definition 4.3 holds for $\overline{J}_{r_{e+1}}$ at every closed point $y \in V(\overline{J}_{r_{e+1}})$, there is a regular system of parameters

$$\{x_1, \dots, x_e, x_{e+1}, \dots, x_d\} \subset \mathcal{O}_{W_{r_e}}$$

such that $\langle x_1, \dots, x_e, x_{e+1} \rangle \subset (\overline{J}_{r_{e+1}})_y$ and if $y \in H_i \in E_{r_{e+1}}$, then $\mathcal{I}(H_i) = \langle x_{i_j} \rangle$ with $i_j > e + 1$. This leads to:

- iv. If the codimension of X in W is $> e + 1$, then $R(e + 1)(V(\overline{J}_{r_{e+1}}))$ must be empty. In fact, (iii) asserts that $R(e + 1)(V(\overline{J}_{r_{e+1}})) \cap V_{r_{e+1}}$ must be dense in $R(e + 1)(V(\overline{J}_{r_{e+1}}))$; but $V_{r_{e+1}}$ is isomorphic to an open subset of W_0 . The claim follows now from (ii), and the assumption that X is of pure codimension $> e + 1$. \square

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